# The evolution of the concepts of energy, momentum, and mass from Newton and Lomonosov to Einstein and Feynman

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# Abstract

These are slides of the talk at the 13th Lomonosov conference, 23 August,2007.

The talk stresses the importance of the concept of rest energy $E_0$ and explains how to use it in various situations.

# 1. Introduction

This conference is the first in a series of conferences celebrating 300 years since the birth of Mikhail Lomonosov (1711–1765).

Therefore it is appropriate to consider the evolution of the laws of conservation of mass, energy, and momentum during this period.

The main message of the talk is the equivalence of the rest energy of a body and its mass: $E_0 = mc^2$. This equivalence is a corollary of relativity principle. The total energy of a body and its mass are not equivalent: $E \neq mc^2$

# 2. Contents

# 3. XVII - XIX centuries

## 3.1. Galileo, Newton: relativiy

The concept of relativity was beautifully described by Galileo Galilei in his famous book “Dialogo” (1632) as experiments in a cabin of a ship.

The principle of relativity had been first formulated by Isaac Newton in his even more famous book “Principia” (1687), though not as a principle, but as corollary v.

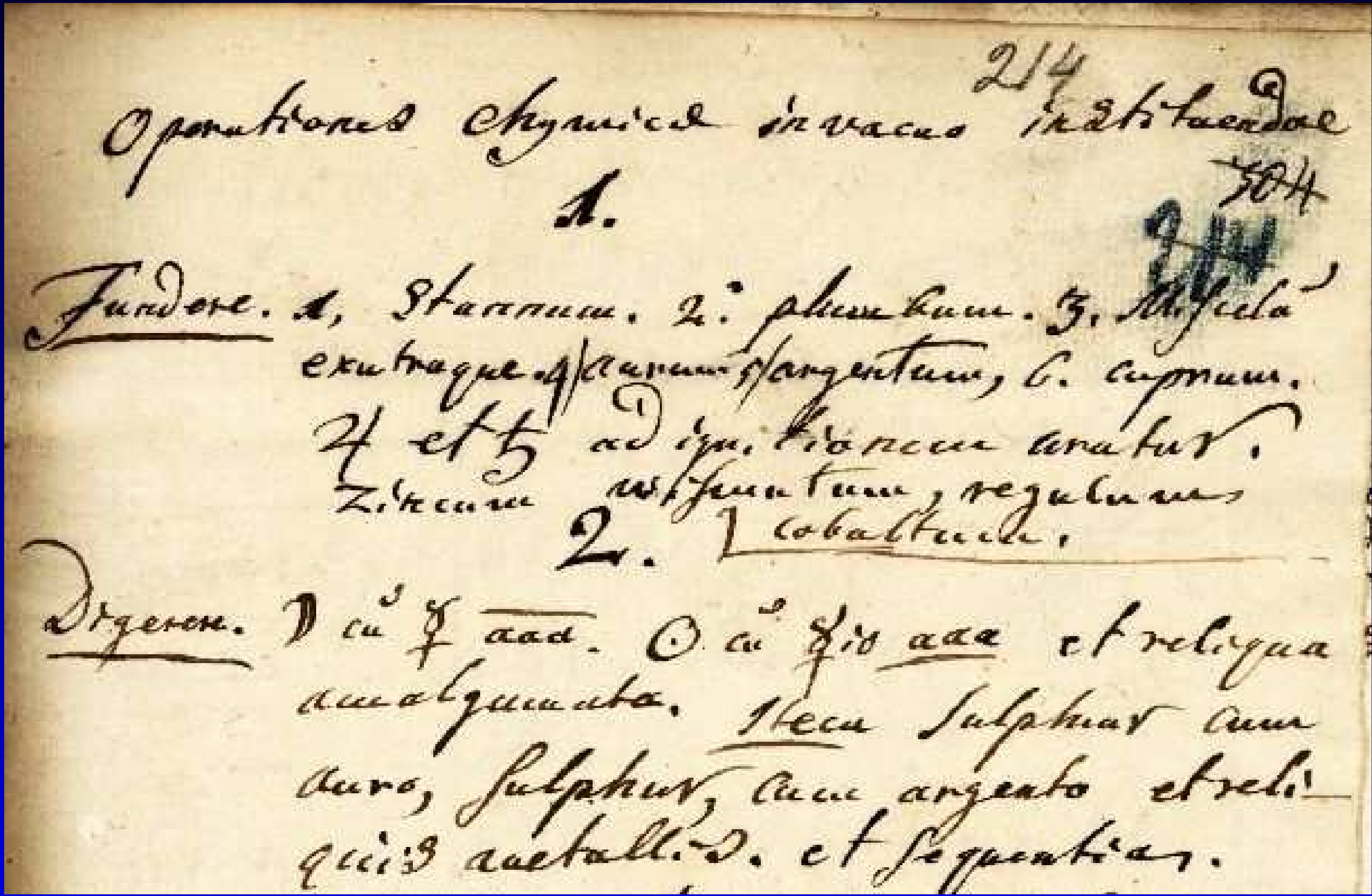
The term mass was introduced into physics by Newton in “Principia”.

According to Newton, the mass is proportional to density and volume. The momentum is proportional to mass and velocity.

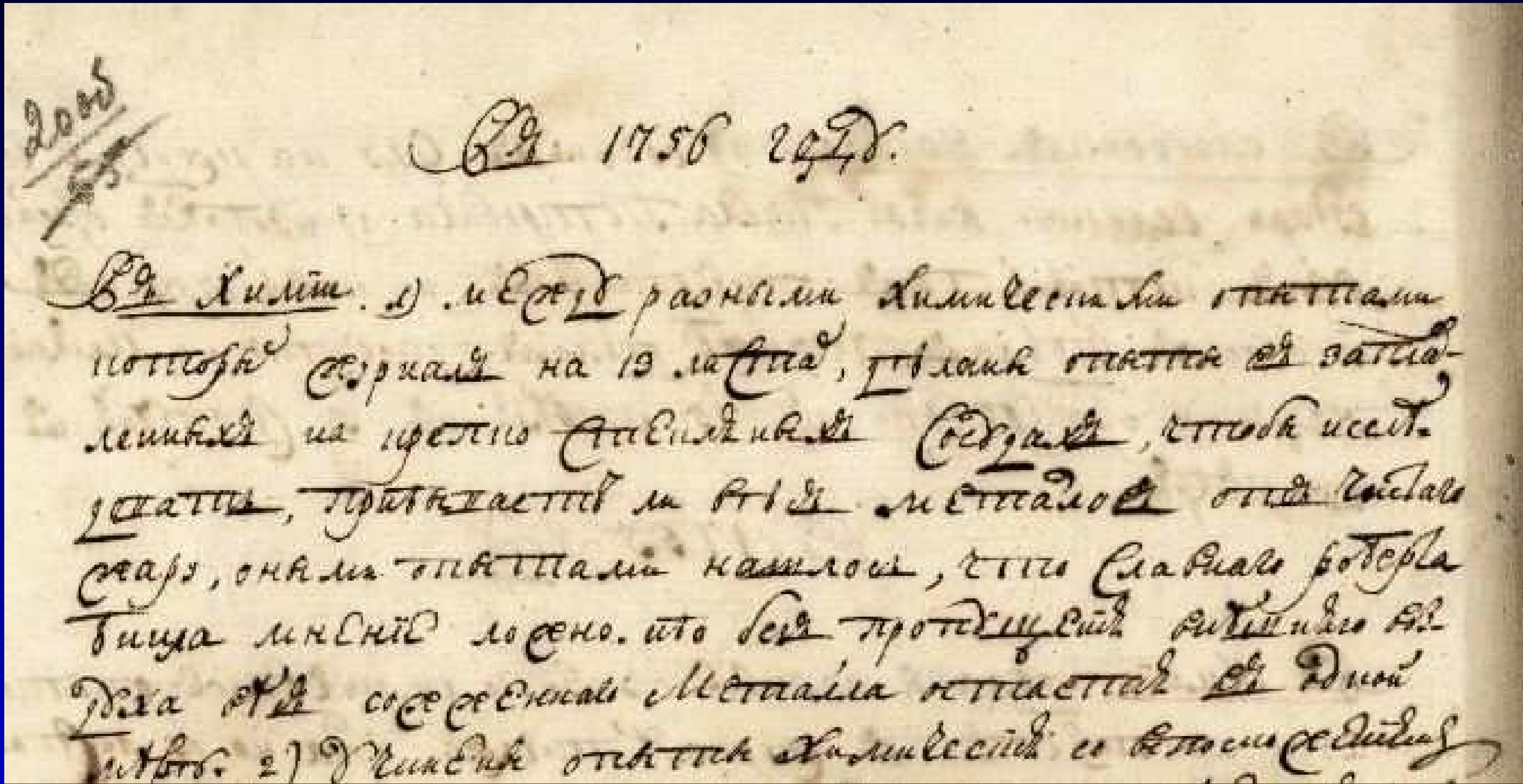
As for the term energy, Newton did not use it. He and Gottfried Leibniz called the kinetic energy vis viva – the living force.

## 3.2. Lomonosov, Lavoisier

In 1756 Lomonosov experimentally proved his earlier conjecture (formulated in his letter to Leonard Euler in 1748) that mass is conserved.



Lomonosov’s handwriting in Latin: ignition of tin (jupiter) and lead (saturnus) in sealed retorts.



В 1756 году.

В химии. 1) между разными химическими опытами, которых журнал на 13 листах, деланы опыты в заплавленных накрепко стеклянных сосудах, чтоб исследовать, прибывает ли вес металлов от чистого жару; оными опытами нашлось, что славного Роберта Бойла мнение ложно, ибо без пропущения внешнего воздуха вес сожженного металла остается в одной мере. 2) Учинены опыты химические с ...

The 1756 report on Lomonosov’s experiments which disproved the results of Robert Boyle on ignition of metals. (Written in Russian by a clerk.)

"... made experiments in firmly sealed glass vessels in order to investigate whether the weight of metals increases from pure heat. It was found by these experiments that the opinion of the famous Robert Boyle is false, for without letting in the external air the weight of the ignited metal remains in the same measure..."

In 1773 Antoine Lavoisier independently proved the law of conservation of mass in a series of more refined experiments.

## 3.4. Conservation of energy

The term energy was introduced into physics in 1807 by Thomas Young.

By the middle of the XIXth century a number of scientists and engineers, especially J.R.von Mayer and J.P. Joule, established the law of conservation of energy which included heat among the other forms of energy.

# 4. The first part of XX century

## 4.1. Rest energy $E_0$

The special theory of relativity was created by Hendrik Lorentz, Henri Poincaré, Albert Einstein, and Herman Minkowski.

The concept of rest energy was introduced into physics by Einstein.

In 1905 Einstein proved in the framework of special relativity that the change of the rest energy of a body is equivalent to the change of its mass.

In 1922 and especially clearly in 1935 he formulated the equivalence of mass $m$ and rest energy $E_0$ – the equation $E_0 = mc^2$.

## 4.2. Energy and inertia

In relativity the energy $E$ and momentum $\boldsymbol{p}$ of a body form the energy-momentum vector $p_i$ $(i = 0, 1, 2, 3 = 0, a)$.

In the units in which $c = 1$: $p_0 = E, p_a = \boldsymbol{p}$.
The mass is a Lorentz scalar
defined by the square of $p_i$:
$m^2 = p^2 = E^2 - \boldsymbol{p}^2$.

To keep track of powers of c let us define
$p_0 = E, p_a = c\boldsymbol{p}$.
Then $p^2 = E^2 - c^2\boldsymbol{p}^2 = m^2c^4$.

In Newtonian physics mass is the measure of inertia according to equations:

$\boldsymbol{p} = m\boldsymbol{v}$ ,

$\boldsymbol{F} = d\boldsymbol{p}/dt$ ,

$\boldsymbol{F} = m\boldsymbol{a}$,

where $\boldsymbol{a} = d\boldsymbol{v}/dt$.

In relativity the energy is the measure of inertia:

$\boldsymbol{p} = E\boldsymbol{v}/c^2$ .

If the force is defined by equation $\boldsymbol{F} = d\boldsymbol{p}/dt$, then

$\boldsymbol{F} = m\gamma\boldsymbol{a} + m\gamma^3\boldsymbol{v}(\boldsymbol{va})$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

In the first years of the XXth century Hendrik Lorentz who tried to use the equation $\boldsymbol{F} = m\boldsymbol{a}$ ended up with the concepts of longitudinal and transverse masses : $m_l = m\gamma^3, m_t = m\gamma$ which later were forgotten.

## 4.3. Energy and gravity

In Newtonian physics the source of gravity is mass. In relativity the source of gravity is the energy-momentum tensor $p_i p_k/E$ which serves as the "gravitational charge".

With the help of propagator of the gravitational field proportional to $g^{ik}g^{lm} + g^{im}g^{lk} - g^{il}g^{km}$, where $g^{ik}$ is the metric tensor, the energy-momentum tensor can be reduced in the static gravitational field (when $k, m = 0$ ) to $(2E^2 - m^2c^4)/E$.

For a massive non-relativistic apple
this expression is equal to $mc^2$,
while for a photon it is equal to $2E$.
Note the factor of 2. The energy of a photon is attracted stronger than the energy of an apple.

## 4.4.“ Relativistic mass” *vs* mass

Three prerelativistic commandments:

1. mass must be the measure of inertia,
2. mass must be conserved,
3. mass must be additive.

They led to the introduction of the so-called “relativistic mass” $m = E/c^2$ which for a massive particle increases with the velocity of the particle.

The idea that mass of an electron increases with its velocity had been put forward by J.J Thomson, O. Heaviside, and G. Sirl in the last decade of the XIXth century, (not so long) before relativity theory was formulated.

The idea that light with energy $E$ has mass $m = E/c^2$ was formulated by Poincaré in 1900 and was discussed by Einstein in the first decade of the XXth century. The relativistic mass increasing with velocity was proclaimed “the mass” by G. Lewis and R. Tolman at the end of that decade. A decade later it was enthroned in books on relativity by Max Born and Wolfgang Pauli.

## 4.5. Famous *vs* true

Thus the equation $E = mc^2$ appeared and was ascribed to Einstein.

This "adopted child" is widely considered as "the famous Einstein's equation" instead of the true Einstein's equation $E_0 = mc^2$. Einstein seemed to be indifferent to this misuse.

## 4.6. Einstein supports $E_0 = mc^2$

In December 1934 Einstein delivered his Gibbs Lecture

“Elementary derivation of the equivalence of mass and energy”

at a joint meeting of the American Mathematical Society and the American Physical Society.

In that lecture he repeatedly stressed that mass $m$ (with the usual time unit, $mc^2$) is equal to rest energy $E_0$.

This however did not prevent Einstein's coauthor – Leopold Infeld* from stating in 1955 that the main experimental confirmation of the special relativity is the dependence of mass on velocity.

* " A.Einstein, L.Infeld. The Evolution of Physics. 1938."

# 5.The second half of XX century

## 5.1. Landau and Lifshitz

The first monograph in which special and general relativity were presented without using the notion of mass increasing with velocity was the first (1941) edition of “Field Theory” by Lev Landau and Evgeniy Lifshitz.

They wrote (in the first and the second editions in §9,§10 –in the later editions they became §8,§9 ) the expressions for action $S$, momentum $\boldsymbol{p}$, energy $E$ and rest energy for which they unfortunately chose the same symbol $E$.

## 5.2. Feynman diagrams

A major step forward in creating the present understanding of nature were diagrams introduced by Richard Feynman.

The external lines of a diagram correspond to incoming and outgoing, free, real particles. For them $p^2 = m^2$ in units of $c = 1$; they are on mass shell.

The internal lines correspond to virtual particles. For them $p^2 \neq m^2$; they are off mass shell.

Energy and momentum are conserved at each vertex of a diagram.

The exchange of a virtual massless particle creates long-range force between real particles.

Thus exchange of a photon creates Coulomb force (potential).

The exchange of a virtual massive particle creates Yukawa potential –short-range force with radius $r = h/mc$.

When using Feynman diagrams, the four-dimensional momenta $p$ and invariant masses $m$ immensely facilitate theoretical analysis of various processes involving elementary particles.

Feynman diagrams unified matter (real particles - both massive and massless) with forces (virtual particles).

The role of Quantum Mechanics is crucial to this unification.

## 5.3. Feynman Lectures

The most famous textbook in physics is "The Feynman Lectures on Physics'.'

Several million copies of Lectures introduced millions of students to physics.

In his Lectures Feynman masterfully and enthusiastically painted the broad canvas of physics from the modern point of view.

Unfortunately in this masterpiece he completely ignored the Feynman diagrams and largely ignored the covariant formulation of the relativity theory.

# 6. Conclusions

The giant figure of Newton marked the birth of modern Science.

The achievements of Science since the times of Newton are fantastic.

The modern views on matter differ drasticlly from those of Newton.

Still, even in the XXIst century many physics textbooks continue to incorrectly use the equations of Newton many orders of magnitude beyond the limits of their applicability, at huge ratios of kinetic energy $E_k$ to rest energy $E_0$ ( $10^5$ for electrons and $10^4$ for protons at CERN), while Newton's equations are valid only for $E_k/E_0 \ll 1$.

If some professors prefer to persist
in this practice, they should at least inform
their students about the fundamental concept of
invariant mass and the true Einstein's equation:

$$E_0 = mc^2$$

# 7. Acknowledgments

I am very grateful for their help to

B.M. Bolotovsky, K.G. Boreskov, M. Gottlieb,

E.G. Gulyaeva, M.V. Danilov, E.A. Ilyina,

O.V. Kancheli, V.I. Kisin, V.I. Kogan,

M.V. Mandrik, B.L. Okun, M.B. Voloshin.

# 8. Discussion: FAQ about mass

## 8.1. Natural definition of mass

Q1: Which definition of mass is natural in the framework of the Relativity Theory?

A1: The definition according to which mass is a Lorentz invariant property of an object – the ‘length’ of the 4-dimensional energy-momentum vector $p = (E, c\boldsymbol{p})$. Namely $m^2 = p^2/c^4$
or in other notations $m^2 = E^2/c^4 - \boldsymbol{p}^2/c^2$.

This definition corresponds perfectly to the fundamental symmetry of special relativity and uses the minimal number of notions and symbols.

## 8.2. Unnatural definition of mass

Q2: Can one nevertheless introduce another definition of mass, namely, that which corresponds to the "famous Einstein's equation $E = mc^2$"?
(here $E$ is the total energy of a free body)

A2: Yes. One can do this. But this cheese is not free. People who do this refer to the ordinary mass as the "rest mass" (they denote it $m_0$).

They have two different symbols for energy: $E$ and $E/c^2 = m$. This is confusing. This ignores the 4-dimensional symmetry of relativity theory: $E$ is a component of a 4-vector, while $E/c^2$ is “the cat that walkes by itself”.

Of course in any consistent theory one can introduce an arbitrary number of redundant variables by multiplying any observable by some power of a fundamental constant, like $c$.

With proper bookkeeping that would not produce algebraic mistakes. However, instead of creating clarity, this creates confusion.

It is like the well known Jewish joke on inserting the letter 'r' in the word 'haim':
– What for is the letter r in the word 'haim'?
– But there is no r in 'haim'
– And if to insert it?
– But what for to insert it?
– That is what I am asking: what for?

Q3: Doesn’t the mass, increasing with the velocity of the body, explain why the velocity of a massive body cannot reach the velocity of light?

A3: No. It does not explain: the increase is not fast enough. This follows from the expression for longitudinal mass $m_l = m_0/(1 - v^2/c^2)^{3/2}$ derived by Lorentz from $\boldsymbol{F} = m\boldsymbol{a}$.

## 8.3. Equivalence of mass and rest energy

Q4: Is mass equivalent to energy?

A4: Yes and no. Loosely speaking, mass and energy are equivalent. But the mass $m$ of an object is not equivalent to its total energy $E$, it is equivalent to its rest energy $E_0$.

Q5: What is rest energy $E_0$?

A5: The rest energy $E_0$ is the greatest discovery of the XXth century.

Einstein discovered that any massive body at rest has a huge hidden energy $E_0 = mc^2$ (the subscript 0 indicates here that the velocity of the body $v$ is equal to zero).

Q5a: How did Einstein discover $E_0 = mc^2$?

A5a: In his second 1905 paper on relativity Einstein considered a body at rest with rest energy $E_0$, which emits two light waves in opposite directions with the same energy $L/2$.

For an observer that moves with velocity $v$ with respect to the body the total energy of two waves is $L/\sqrt{1 - v^2/c^2}$.

By assuming conservation of energy and by considering the case of $v \ll c$ Einstein derived that $\Delta m = L/c^2$.

In this short note two revolutionary ideas were formulated:
1. that a massive body at rest contains rest energy $E_0$,
2. that a system of two massles light waves with energy $L$ has mass $L/V^2$. (Einstein denoted the speed of light by $V$.)

In his publications of 1922 and 1935 Einstein cast the relation in the form $E_0 = mc^2$.

Q5b: Is it possible to prove Einstein's relation by considering emission of one wave of light instead of two?

A5b: Yes, it is possible. But the proof is slightly more involved. In this case the rest energy of the body partly transforms into kinetic energy of light $L$ and kinetic energy of the recoil body with mass m: $E_k = L^2/2mc^2$.

Q6: Is the relation $E_0 = mc^2$ compatible with the definition of mass given above: $m^2 = E^2/c^4 - \boldsymbol{p}^2/c^2$ ?

A6: Yes. It is absolutely compatible: at $\boldsymbol{v} = 0$ you have $\boldsymbol{p} = 0$, while $E = E_0$.

Q6a: Why not abandon the term “mass” in favor of “ rest energy”? Why to have two terms instead of one, if we do know that mass is equivalent to rest energy?

A6a: “One” is not always better than “two”. The word “mass” refers to a lot of phenomena which have nothing to do with the rest energy “sleeping” in massive bodies. Such a terminological reform would be a disaster not only for Newtonian mechanics for which $c$ is alien, but for Science in general.

Q6b: Isn't it better to have both relations: $E_0 = mc^2$ and $E = mc^2$ instead of one of them? Isn't "two " always better than "one"? Recall the famous wave-particle duality.

A6b: Two relations (explanations) are better than one if both are correct and if each of them has its own realm of applicability. The relation $E = mc^2$ has no separate domain of applicability. Moreover it has no domain of applicability at all. It is a consequence of introduction, along with $E$, of a redundant variable of "relativistic mass" $E/c^2$ which usurped the throne of mass. Thus in this case "one" is much better than "two".

Q6c: Why do you dislike the relativistic mass so strongly?

A6c: I stumbled on it 20 years ago and realized how difficult it is to reeducate students and teachers brought up on the concept of mass increasing with velocity and the famous formula $E = mc^2$. It selfpropagates like a virus or a weed and prevents people from understanding the essence of relativity theory.

A century ago Max Planck said that the carriers of wrong views simply die out while new generations accept the truth. But it turned out that new generations come already infected. An important role in the mechanism of infection was played the authors of textbooks and popular science writers, the editors of popular magazines, like “Scientific American”. It is the rare case when most of the experts know the truth, but lightly preach the non-truth.

Q7: Does the mass of a box filled with gas increase with the increase of the temperature of the gas?

A7: Yes, according to relativity theory, it increases.

Q8: Doesn't it mean that the masses of molecules of gas increase with temperature, i.e. with their velocities? Or in other words, that energy and mass are equivalent?

A8: No, it does not mean that. Such an inference would presume the additivity of masses. But according to relativity theory, the total mass of the gas is not equal to the sum of the masses of its molecules: $m \neq \sum m_i$.

In fact the correct interpretation of the mass of the gas supports the relation $E_0 = mc^2$, not the relation $E = mc^2$.

This can be seen from the following reasoning. The total energy $E$ of the relativistic gas is equal to the the sum of total energies $E_i$ of the individual particles of gas: $E = \sum E_i$. Each $E_i$ increases with temperature. Hence the total energy of gas increases.

The total momentum $\boldsymbol{p}$ of the gas vanishes because the distribution of particle's momenta $\boldsymbol{p}_i$ is isotropic: $\boldsymbol{p} = \sum \boldsymbol{p}_i = 0$. Hence the total energy of gas is equal to its rest energy.

By applying the definition of invariant mass: $m^2 = E^2/c^4 - \boldsymbol{p}^2/c^2$ and taking into account that in this case $E = E_0$ one gets $E_0 = mc^2$.

This is valid both for the gas of massive particles and for massless photons.

## 8.4. Interconversion between rest energy and kinetic energy

Q9: Does mass convert into energy? Does energy convert into mass?

A9: No. The “mutual conversion of mass and energy” is a very loose and therefore a misleading term. The point is that energy is strictly conserved in all processes. It can neither appear, nor disappear. It can only transform from one form into another. The rest energy (mass) converts into other forms of energy (e.g. kinetic energy).

Q10: Does energy convert into mass in the processes of production of particles in accelerators?

A10: No. Various forms of energy transform into each other, but the total energy is conserved.

The kinetic energy transforms into rest energy (into masses of the produced particles) in accelerators. The colliders convert $E_k$ into mass much more effectively than the fixed target accelerators.

Q11: Did the laws of conservation of mass and energy merge into one law of conservation of mass-energy similar to the law of conservation of energy-momentum 4-vector?

A11: No. The laws of conservation of energy and momentum of an isolated system (unified in the law of conservation of 4-momentum) correspond to the uniformity of time and space correspondingly. There is no extra space-time symmetry responsible for the conservation of mass.

The total mass of a closed (isolated) system (the rest energy of the system) is conserved due to conservation of its energy and momentum.

Q12: Doesn’t the total mass change in the annihilation of positronium into two photons? Electron and positron are massive, while photons are massless.

A12: No. The total mass does not change: the rest energy of the system of two massless photons is equal, in this process, to the rest energy of positronium.

Q12a: What is the meaning of the term “rest energy of the system of two photons”, if each of them has no rest energy and in a second after they were born the two photons are 600 000 km apart?

A12a: “Rest energy of the system of two photons” means here the sum of their kinetic (or total) energies in a reference frame in which the sum of their momenta is equal to zero. In this frame they fly in opposite directions with equal energies.

Q12b: Why do you refer to this rest energy as mass of the system of two photons?

A12b: Because I am applying the equation $E_0 = mc^2$.

The mass of an elementary particle has a deep physical meaning because it is an important quantum number characteristic of all elementary particles of a given sort (say, electrons or protons). The mass of a nuclear or atomic level is also a quantum number.

The mass of a macroscopic body is not as sharply defined because of overlap of huge number of quantum levels.

As for the mass of a system of free particles, it is simply their total energy (divided by $c^2$) in a frame in which their total momentum is equal to zero. The value of this mass is limited only by conservation of energy and momentum, like in the case of two photons in the decay of positronium.

As a rule we are unable to measure the inertia or gravity of such a system, but the self-consistency of the relativity theory guarantees that it must behave as mass

Q12c: Do I understand correctly that with this definition of mass the conservation of mass is not identical to the conservation of matter in the sense in which it was meant by Lomonosov and Lavoisier?

A12c: Yes. You do understand correctly. Matter now includes all particles,even very light neutrinos and massless photons. The number of particles in an isolated piece of matter is not conserved. Roughly speaking, the mass of a body is a sum of masses of constituent particles plus their kinetic energies minus the energy of their attraction to each other ( of course, the energies are divided by $c^2$).

## 6.5. Binding energy in nuclei

Q13: Is the mass of a nucleus equal to the sum of the masses of the constituent nucleons?

A13: No. The mass of a nucleus is equal to the sum of the masses of the constituent nucleons minus the binding energy divided by $c^2$.

Thus the nucleus is lighter than the sum of the masses of its nucleons.

Q14: Can the liberation of kinetic energy in the Sun, in nuclear reactors, atomic and hydrogen bombs be explained without referring to the equation $E_0 = mc^2$?

A14: Yes. In the same way that it is explained for chemical reactions, namely, by the existence and difference of binding energies.

Rutherford considered the dependence of mass on velocity as an important fact, but neither he nor his coworkers mentioned $E = mc^2$ or $E_0 = mc^2$ in their works as a source of energy released in radioactive processes though they rejected the idea of perpetum mobile.

## 8.6. Mass differences of hadrons

Q15: Is the mass of a proton equal to the sum of the masses of two $u$ quarks and one $d$ quark which constitute the proton?

A15: No. The mass of the proton is not equal to the sum of the masses of three quarks. However, the situation here is more subtle than in the case of nucleons in a nucleus.

Q16: What is the main difference between quarks and nucleons?

A16: Nucleons can exist as free particles.( Hydrogen is the most abundant element in the universe, while free neutrons are produced in nuclear reactors.)

Quarks exist only inside hadrons.
Free quarks do not exist.
Mass is defined by equation $m^2 = E^2/c^4 - p^2/c^2$ only for free particles.
Therefore, strictly speaking, we cannot apply this equation to quarks.
However one can use the property of asymptotic freedom of QCD – Quantum Chromodynamics.

Q17: What is asymptotic freedom?

A17: According to the asymptotic freedom, the higher the momentum transfer is in interaction of quarks, the weaker their interaction is.

Thus, due to the uncertainty relation, at very short distances quarks look like almost free particles.

In units where $c = 1$ the mass of $u$ quark is 4 MeV at such distances,while that of $d$ quark is 7 MeV. The sum of masses of three quarks inside a proton is 15 MeV, while the mass of the proton is 938 MeV.

Q18: What constitutes the difference between 938 MeV and 15 MeV?

A18: This difference – the main part of the proton mass, as well as of the masses of other hadrons – is caused mainly by the energy of the gluon field – the vacuum condensate of gluons.

Q19: Can we speak about the values of this condensate as of binding energies?

A19: No, we cannot. The contribution of binding energy to the mass is negative, while the contribution of condensate is positive.

By supplying enough energy from outside one can liberate a nucleon from a nucleus, but one cannot liberate a quark in that way from the confinement inside a hadron.

Q20: Can we understand the source of the kinetic energy in beta decay of the neutron without invoking $E_0 = mc^2$?

A20: No, we cannot. Because we cannot express the mass difference between a neutron and a proton in terms of binding energies as we did for nuclei. This is even more so for lepton masses.

## 8.7. Some basic questions

Q21: Why does the velocity of light $c$ enter the relation between the mass and the rest energy?

A21: Because $c$ is not only the velocity of light but also the maximal speed of propagation of any signal in Nature.

As such, it enters all fundamental interactions in Nature as well as Lorentz transformations.

Q22: Why do you claim that gravity is reducible to the interaction of energies, not masses?

A22: Because a massless photon is attracted by the gravitational field of the Sun. (The deflection of light was first observed in 1919 and brought Einstein world fame.) As for the massive particle, its mass is equal to its rest energy. Thus in both cases we deal with energy.

There is also another argument in favor of energy as a source of gravity.

I refer here to the fact established by Galileo almost four centuries ago and cofirmed in the XXth century with accuracy $10^{-12}$. Namely, that all bodies have the same gravitational acceleration. It does not depend on their composition, on the proportions between different terms in their rest energy. That means that only the total rest energy of a slow body determines both its gravitational attraction and its inertia.

Q23: How was this fact explained in the framework of prerelativistic physics and how is it explained by relativity theory ?

A23: In the prerelativistic physics it was formulated as a mysterious equality of inertial mass $m_i$ and gravitational mass $m_g$.

In relativity theory it became trivial, because both inertia and gravity of a body are proportional to its total energy.

Q24: What are the main directons in the research on the concept of mass in the next decade?
A24: The main experimental direction is the search for higgs at LHC at CERN. According to the Standard Model, this particle is responsible for the masses of leptons and quarks as well as of $W$ and $Z$ bosons. Of great interest is also the experimental elucidation of the pattern of neutrino masses and mixings.

The main cosmological direction is the study of dark matter and dark energy.

Q25 : What was the formulation of the Corollary v in "The Principia"?

A25: Here is the citation from "The Principia":

Sir Isaac Newton. The Principia. Axioms, or Laws of Motion.

COROLLARY V. The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.

For the differences of the motions tending towards the same parts, and the sums of those that tend towards contrary parts, are, at first (by supposition), in both cases the same; and it is from those sums and differences that the collisions and impulses do arise with which the bodies mutually impinge one upon another. Wherefore (by Law II), the effects of those collisions will be equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the mutual motions of the bodies among themselves in the other. A clear proof of which we have from the experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line.