UNSTABLE LEPTONS AND $(\mu - \mathrm{e} - \tau)$-UNIVERSALITY

O. S. Kosmachev

Joint Institute for Nuclear Research,
Laboratory of High Energies,
141980, Moscow Region, Dubna, Russia

Main advantage and virtue of proposed method is a possibility to describe and enumerate

all possible types of free equations for stable and unstable leptons

in the frame work of homogeneous Lorentz group by means of unique approach.

WHY IT IS NECESSARY?

1. Free states are necessary for **description of interactions**. As it is known they play the role of initial and final states.
2. Free states equations are unique way to introduce in theory quantum numbers identifying any leptons. These quantum numbers characterize an equation structure and will be called **structural quantum numbers**.

The proposed method succeed from those fundamental requirements as Dirac equation (1928):

1. Invariance of the equations relative to homogeneous Lorentz group taking into account four connected components.
2. Formulation of the equations on the base of irreducible representations of the groups, determining every lepton equation.
3. Conservation of four-vector of probability current and positively defined fourth component of the current.
4. Spin value of the leptons is proposed equal to 1/2.

One can show that a totality of enumerated physical requirements are necessary and sufficient conditions (together with some group-theoretical requirements) for formulation of lepton wave equation out of Lagrange formalism.

Dirac equation and discrete symmetries

One can show (Kosmachev, 2004), that Dirac equation is related with three different irreducible representations of homogeneous Lorentz group.It follows from the fact that Dirac γ-matrix group contains two subgroups d_γ, b_γ and dual property of d_γ.
In this case

1. standard (proper, orthochronous) representation is realized on d_γ group,
2. T-conjugate representation is realized on b_γ group,
3. P-conjugate representation is realized on f_γ group,

Corresponding algebras (six-dimensional Lie algebras of homogeneous Lorentz group) are characterized completely by their commutative relations. They are of the form for d_γ group

$$\begin{aligned} &[a_i, a_k] = \varepsilon_{ikl} 2a_l, \\ &[b_i, b_k] = -\varepsilon_{ikl} 2a_l, \qquad \varepsilon_{ikl} = \begin{cases} 1 \\ -1 \\ 0 \end{cases} \\ &[a_i, b_k] = \varepsilon_{ikl} 2b_l. \end{aligned}$$

Here ε_{ikl} is Levi-Cevita tensor, $i, k, l = 1, 2, 3$ a_i, b_i are infinitesimal operators of three-rotations and boosts respectively.

Commutative relations(CR) on the base of d_γ:

$$\begin{array}{lll}
[a_1,a_2]=2a_3, & [a_2,a_3]=2a_1, & [a_3,a_1]=2a_2,\\
[b_1,b_2]=-2a_3, & [b_2,b_3]=-2a_1, & [b_3,b_1]=-2a_2,\\
[a_1,b_1]=0, & [a_2,b_2]=0, & [a_3,b_3]=0,\\
[a_1,b_2]=2b_3 & [a_1,b_3]=-2b_2, & \\
[a_2,b_3]=2b_1, & [a_2,b_1]=-2b_3, & \\
[a_3,b_1]=2b_2, & [a_3,b_2]=-2b_1. &
\end{array}$$

where: $a_1 \sim \gamma_3\gamma_2$, $a_2 \sim \gamma_1\gamma_3$ $a_3 \equiv a_1a_2 \sim \gamma_2\gamma_1$, $a_2a_1a_2^{-1}=a_1^{-1}$, $b_1 \sim \gamma_1$, $b_2 \sim \gamma_2$, $b_3 \sim \gamma_3$. Here following definitions are used

$$[i(\gamma_\mu p_\mu)+mc]\Psi=0,$$
$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu},\quad \mu,\nu=1,2,3,4.$$

(Dirac P., Proc.Roy.S. 1928).
Commutative relations on the base of b_γ:

$$\begin{array}{lll}
[a_1,a_2]=2a_3, & [a_2,a_3]=2a_1, & [a_3,a_1]=2a_2,\\
[b'_1,b'_2]=2a_3, & [b'_2,b'_3]=2a_1, & [b'_3,b'_1]=2a_2,\\
[a_1,b'_1]=0, & [a_2,b'_2]=0, & [a_3,b'_3]=0,\\
[a_1,b'_2]=2b'_3 & [a_1,b'_3]=-2b'_2, & \\
[a_2,b'_3]=2b'_1, & [a_2,b'_1]=-2b'_3, & \\
[a_3,b'_1]=2b'_2, & [a_3,b'_2]=-2b'_1, &
\end{array}$$

where;$b'_1 \equiv c'a_1 \sim -\gamma_1\gamma_4$, $b'_2 \equiv c'a_2 \sim -\gamma_2\gamma_4$, $b'_3 \equiv c'a_3 \sim -\gamma_3\gamma_4$, $c'=a_3b_5$.
Subgroups d_γ and b_γ have different structures therefore impossible to express one system of CR via another by means of nonsingular transformations.

Commutative relations on the base of f_γ-group:

$$
\begin{array}{lll}
[a_1, a_2'] = 2a_3', & [a_2', a_3'] = -2a_1, & [a_3', a_1] = 2a_2', \\
[b_1', b_2'] = -2a_3', & [b_2', b_3'] = 2a_1, & [b_3', b_1'] = -2a_2', \\
[a_1, b_1'] = 0, & [a_2', b_2'] = 0, & [a_3', b_3'] = 0, \\
[a_1, b_2'] = 2b_3', & [a_1, b_3'] = -2b_2', & \\
[a_2', b_3'] = -2b_1', & [a_2', b_1'] = -2b_3', & \\
[a_3', b_1'] = 2b_2', & [a_3', b_2'] = 2b_1'. &
\end{array}
$$

If to construct an algebra on c_γ, we obtain commutative relations:

$$
\begin{array}{lll}
[a_1, a_2'] = 2a_3', & [a_2', a_3'] = -2a_1, & [a_3', a_1] = 2a_2', \\
[b_1'', b_2''] = 2a_3', & [b_2'', b_3''] = -2a_1, & [b_3'', b_1''] = 2a_2', \\
[a_1, b_1''] = 0, & [a_2', b_2''] = 0, & [a_3', b_3''] = 0, \\
[a_1, b_2''] = 2b_3'' & [a_1, b_3''] = -2b_2'', & \\
[a_2', b_3''] = -2b_1', & [a_2', b_1''] = -2b_3'', & \\
[a_3', b_1''] = 2b_2'', & [a_3', b_2''] = 2b_1''. &
\end{array}
$$

Now we have the complete set of constituents for description of lepton wave equations.

Equations for stable leptons

The base of every lepton equation is a corresponding γ-matrix group. Each of five γ-matrix group are produced by **four** generators. **Three** of them anticommute and ensure Lorentz invariance of different kinds. The fourth generator is a necessary condition for the formation of wave equation. The distinct nonidentical equations are became by virtue of different combinations of the four subgroups $d_\gamma, b_\gamma, c_\gamma, f_\gamma$. Structural consistent and determining relations for every lepton equation are reduced in explicit form.

Structural content of the groups for every type of equation has the form.

1. Dirac equation — $D_\gamma[II]$: $d_\gamma, b_\gamma, f_\gamma$.
2. Equation for doublet massive neutrino — $D_\gamma[I]$: $d_\gamma, c_\gamma, f_\gamma$.
3. Equation for quartet massless neutrino — $D_\gamma[III]$: $d_\gamma, b_\gamma, c_\gamma, f_\gamma$.
4. Equation for massless T-singlet — $D_\gamma[IV]$: b_γ.
5. Equation for massless P-singlet — $D_\gamma[V]$: c_γ.

Corollaries.

1. Every equation has its own structure allowing to distinguish one equation from other.
2. All equations have not physical substructures, therefore leptons are stable.
3. Obtained method allows to calculate full number of leptons in the framework of starting suppositions.

EXTENSIONS OF THE STABLE LEPTON GROUPS

Is it possible to obtain additional lepton equations on the bas of previous suppositions?

YES.

This problem is attained by introducing additional (fifth) generator for new group production.As it turned out there are exist three and only three such possibilities. Each of them is equivalent to introduction of additional quantum characteristics (quantum numbers) .

1. The extension of Dirac γ-matrix group $(D_\gamma(II))$ by means of anticommuting generator Γ_5 such that $\Gamma_5^2 = I$ leads to Δ_1-group with structural invariant equal to $In[\Delta_1] = -1$.
2. The extension of Dirac γ-matrix group by means of anticommuting generator Γ'_5 such that ${\Gamma'_5}^2 = -I$ leads to Δ_3-group with structural invariant equal to $In[\Delta_3] = 0$.
3. The extension of neutrino doublet group $(D_\gamma(I))$ by means of anticommuting generator Γ''_5 such that ${\Gamma''_5}^2 = -I$ leads to Δ_2-group with structural invariant equal to $In[\Delta_2] = 1$.

$\mathbf{\Delta_1}$-group has the following defining relations

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu}, \quad (\mu, \nu = 1, 2, 3, 4, 5) \tag{1}$$

It follows from them

$$\Gamma_6 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5, \quad \Gamma_6 \Gamma_\mu = \Gamma_\mu \Gamma_6, \quad (\mu, \nu = 1, 2, 3, 4, 5) \tag{2}$$

It means that Γ_6 belong to group center and $\Gamma_6^2 = I$.
One can show on the bas of (1) that Δ_1 contains 3 and only 3 subgroups of 32-order.
As a result we have following content

$$\Delta_1 \{D_\gamma(II), \quad D_\gamma(III), \quad D_\gamma(IV)\} \tag{3}$$

Relation (3) together with structural invariant $In[\Delta_1] = -1$ identify Δ_1 in physical sense.

$\mathbf{\Delta_3}$-group is obtained under extension of Dirac group by similar defining relations

$$\begin{array}{ll} \Gamma_s\Gamma_t + \Gamma_t\Gamma_s = 2\delta_{st}, & (s,t = 1,2,3,4), \\ \Gamma_s\Gamma_5 + \Gamma_5\Gamma_s = 0, & (s = 1,2,3,4), \\ \Gamma_5^2 = -1. & \end{array}$$

It follows that

$$\Gamma_6 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5, \quad \Gamma_6\Gamma_\mu = \Gamma_\mu\Gamma_6, \quad (\mu = 1,2,3,4,5). \tag{4}$$

As in previous case Γ_6 belong to group center and $\Gamma_6^2 = -I$. It means in matrix realization $\Gamma_6 = \pm iI$.
The group content was changed in this way

$$\Delta_3\{D_\gamma(II), \quad D_\gamma(I), \quad D_\gamma(III)\}, \tag{5}$$

This corresponds to structural invariant $In[\Delta_3] = 0$.

$\mathbf{\Delta_2}$-group and it defining relations.

$$\begin{array}{ll} \Gamma_s\Gamma_t + \Gamma_t\Gamma_s = 2\delta_{st}, & (s,t = 1,2,3), \\ \Gamma_s\Gamma_4 + \Gamma_4\Gamma_s = 0, & (s = 1,2,3), \\ \Gamma_4^2 = -1. & \\ \Gamma_u\Gamma_5 + \Gamma_5\Gamma_u = 0, & (u = 1,2,3,4), \\ \Gamma_5^2 = -1. & \end{array}$$

Consequently

$$\Gamma_6 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5, \quad \Gamma_6\Gamma_\mu = \Gamma_\mu\Gamma_6, \quad (\mu = 1,2,3,4,5) \tag{6}$$

Γ_6 belong to group center and $\Gamma_6^2 = I$.
The group content differs from two previous cases

$$\Delta_2\{D_\gamma(I), \quad D_\gamma(III), \quad D_\gamma(V)\}, \tag{7}$$

Structural invariant is equal to $In[\Delta_2] = 1$.

CONCLUSION

All examined equations have its own

mathematical structure .

These structures are not repeated, therefore they may be used for theoretical identification of the particles in free states. The first five equations including Dirac one

have not physical substructures .

Objects without structure can not disintegrate spontaneously , therefore all they are

stable .

The last three equations ($\mathbf{\Delta_1}, \mathbf{\Delta_2}, \mathbf{\Delta_3}$) have internal structures allowing of physical interpretation. If we suppose that the mass of the new particles is more than sum of masses of its constituents, they become candidates for

unstable leptons .

It is evidentally that equations on the base of $\mathbf{\Delta_1}$ and $\mathbf{\Delta_3}$ may be interpreted as the equations for the massive charged leptons such as $\mu^{\pm}$ and $\tau^{\pm}$. It is possible to relate $\mathbf{\Delta_2}$-group with **massive unstable neutrino**.

APPENDICES

The wave equation for Δ_1 is formulated by analogy with Dirac eqation

$$[i\sum_{a=1}^{4}(\Gamma_a p_a)+\Gamma_6 m]\psi=0,\quad \Gamma_6=\pm I \tag{8}$$

$$\Gamma_1=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\Gamma_2=\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\Gamma_3=\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\Gamma_4=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\Gamma_5=\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

The wave equation for Δ_3 has the form

$$[\sum_{a=1}^{4}(\Gamma_a p_a) \pm m]\psi = 0, \quad \Gamma_6 = \pm iI,$$

where

$$\Gamma_1 = \left(\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right), \Gamma_2 = \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right), \Gamma_3 = \left(\begin{array}{cccc} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{array}\right),$$

$$\Gamma_4 = \left(\begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \Gamma_5 = \left(\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right),$$

The wave equation on the base of Δ_2-group has the form

$$[i\sum_{a=1}^{4}(\Gamma_a p_a)+\Gamma_6 m]\psi=0, \quad \Gamma_6=\pm I,$$

where

$$\Gamma_1=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \Gamma_2=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \Gamma_3=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_4=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \Gamma_5=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

All matrices are real. It is corollary of the property $In[\Delta_2]=1$.

A new and effective tool for analysis and constructing lepton equations was found, i.e. numerical characteristic of irreducible matrix group.

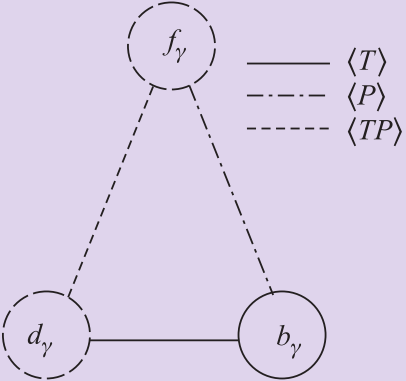
Theorem. If $\Gamma = \{\gamma_1, ..., \gamma_\rho\}$ is an irreducible matrix group, then

$$\mathbf{In} = \frac{\mathbf{1}}{\rho} \sum_{\mathbf{i=1}}^{\rho} \chi(\gamma_{\mathbf{i}}^{\mathbf{2}}) = \begin{cases} 1 \\ -1 \\ 0 \end{cases} \tag{9}$$

Here ρ - is order of the group, $\chi(\gamma_i^2)$ - is a trace of i-matrix squared.
$\mathbf{In}$ — will be called **structural invariant**.

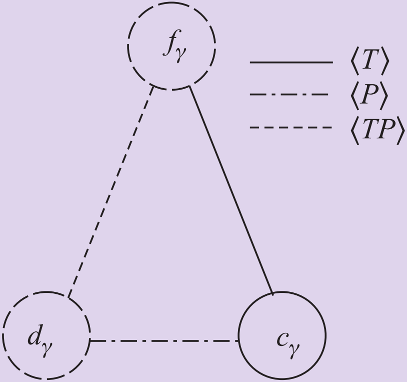
Dirac equation (doublet e^+e^-)

$D_\gamma(II) : d_\gamma, b_\gamma, f_\gamma.$
$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu},$
$\mu, \nu = 1, 2, 3, 4.$
$[i(\gamma_\mu p_\mu) + m]\,\Psi(x,t) = 0$
$In[D_\gamma(II)] = -1$



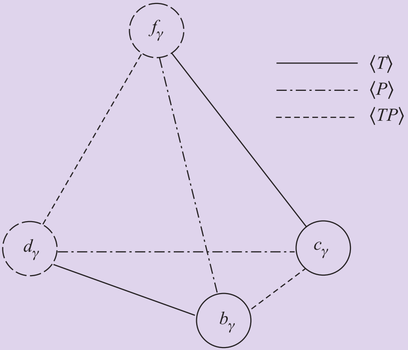
Neutrino doublet

$D_\gamma(I) : d_\gamma, c_\gamma, f_\gamma.$
$\gamma_s\gamma_t + \gamma_t\gamma_s = 2\delta_{st},$
$\gamma_4\gamma_s + \gamma_s\gamma_4 = 0$
$\gamma_4^2 = -1, s, t = 1, 2, 3$
$[i(\gamma_\mu p_\mu) - m]\,\Psi(x, t) = 0$
$In[D_\gamma(I)] = 0$



Neutrino quartet

$D_\gamma(III) : d_\gamma, b_\gamma, c_\gamma, f_\gamma.$
$\gamma_s\gamma_t + \gamma_t\gamma_s = 2\delta_{st},$
$\gamma_4\gamma_s - \gamma_s\gamma_4 = 0$
$\gamma_4^2 = 1, s, t = 1, 2, 3$
$(\gamma_s p_s)\Psi(x,t) - \gamma_4\partial\Psi(x,t)/\partial t = 0,$
$\mathrm{In}[D_\gamma(III)] = 1$



Singlets(absolutely neutral particles)

T-singlet

$D_\gamma(IV)$**:** b_γ**.**

$\gamma_s\gamma_t + \gamma_t\gamma_s = -2\delta_{st}$, s,t=1,2,3

$\gamma_4\gamma_s - \gamma_s\gamma_4 = 0$, $\gamma_4^2 = 1$,

$(ip_4\gamma_4 - p_1\gamma_1 - p_2\gamma_2 - p_3\gamma_3)\Psi(\mathbf{x}, t) = 0$

$\mathbf{In}[D_\gamma(IV)] = -1$

P-singlet

$D_\gamma(V)$**:** c_γ**.**

$\gamma_s\gamma_t + \gamma_t\gamma_s = 0,$, $s \neq t$, s,t=1,2,3

$\gamma_1^2 = \gamma_2^2 = 1, \gamma_3^2 = -1,$

$\gamma_4\gamma_s - \gamma_s\gamma_4 = 0$, $\gamma_4^2 = 1$,

$(p_4\gamma_4 - ip_1\gamma_1 - p_2\gamma_2 - p_3\gamma_3)\Psi(\mathbf{x}, t) = 0,$

$\mathbf{In}[D_\gamma(V)] = 1$

Here $p_4 = -i\partial/\partial t$, $p_s = -i\partial/\partial x_s, \quad s = 1, 2, 3.$