Algebraic approach to analytical evaluations of Feynman diagrams

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Plan

1. **Motivation**
2. **The diagrams** ↔ Perturbative integrals
   - Which kind of Feynman diagrams (F.D.) we consider
3. **Operator formalism**
   - Algebraic reformulation of integrals for F.D.: manipulations with integrals → manipulations with operators
4. **Application**
   - Ladder diagrams for ϕ^3-theory in $D = 4$; relations to conformal quantum mechanics

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- In perturbative QFT physical data are extracted from multiple integrals (perturbative integrals) associated to F.D.
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2. The diagrams

The F.D. (considered here) are graphs with vertices connected by lines labeled by numbers (indeces).

To each vertex of the graph we associate the point in D-dimensional Euclidean space $\mathbf{R}^D$, while the lines (edges) of the graph (with index α) are propagators of massless particles

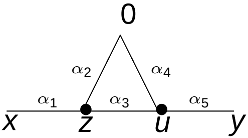
$$x \quad \overset{\alpha}{\overline{\hspace{3cm}}} \quad y \qquad = \qquad 1/(x-y)^{2\alpha}$$

where $\boxed{(x-y)^{2\alpha} := (\sum_{i=1}^{D}(x_i - y_i)\,(x_i - y_i))^{\alpha}}$, $\alpha \in \mathbf{C}$, $x, y \in \mathbf{R}^D$. We have 2 types of vertices: the boldface vertices $\bullet$ denote the integration over $\mathbf{R}^D$. These F.D. are called F.D. in the configuration space.

2. The diagrams

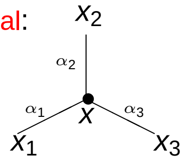
Examples (F.D. in configuration space):

a. 3-point function (graph with 5 vertices and 5 edges):



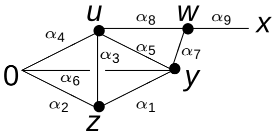
$$= \int \frac{d^Dz\, d^Du}{(z-y)^{2\alpha_1}\, z^{2\alpha_2}\, y^{2\alpha_3}\, u^{2\alpha_4}\, (u-y)^{2\alpha_5}}$$

b. Star integral:



$$= \int \frac{d^Dx}{(x-x_1)^{2\alpha_1}\, (x-x_2)^{2\alpha_2}\, (x-x_3)^{2\alpha_3}}$$

c. Propagator-type diagram:



$$= \int \frac{d^Dz\, d^Du\, d^Dy\, d^Dw}{(x-z)^{2\alpha_1}\, z^{2\alpha_2}\, (z-u)^{2\alpha_3}\, u^{2\alpha_4}\, (u-y)^{2\alpha_5}\, y^{2\alpha_6} \ldots (w-x)^{2\alpha_9}}$$

Analytical calc. of F.D. → reconstruction of graphs to reduce no. of •.

3. Operator formalism

Consider D-dimensional Euclidean space $\mathbf{R}^D$ with coordinates x_i, $(i=1,2,\dots,D)$. We use notation: $\boxed{x^{2\alpha}=(\sum_{i=1}^{D}x_i^2)^\alpha}$. Let $\hat{q}_i=\hat{q}_i^\dagger$ and $\hat{p}_i=\hat{p}_i^\dagger$ be operators of coordinate and momentum

$$\boxed{[\hat{q}_k\,,\,\hat{p}_j]=\mathrm{i}\,\delta_{kj}}\;.$$

Introduce states $|x\rangle\equiv|\{x_i\}\rangle$, $|k\rangle\equiv|\{k_i\}\rangle$: $\hat{q}_i|x\rangle=x_i\,|x\rangle$, $\hat{p}_i|k\rangle=k_i\,|k\rangle$, and normalize these states as:

$$\langle x|k\rangle=\frac{1}{(2\pi)^{D/2}}\,\exp(\mathrm{i}\,k_j\,x_j)\;,\quad \int d^Dk\,|k\rangle\,\langle k|=\hat{1}=\int d^Dx\,|x\rangle\,\langle x|\;.$$

"Matrix representation" of $\hat{p}^{-2\beta}$ (propagator of massless particle) is:

$$\underline{\langle x|\frac{1}{\hat{p}^{2\beta}}|y\rangle=a(\beta)\,\frac{1}{(x-y)^{2\beta'}}}\;,\qquad \left(a(\beta)=\frac{\Gamma(\beta')}{\pi^{D/2}\,2^{2\beta}\,\Gamma(\beta)}\right)\,.$$

where $\boxed{\beta'=D/2-\beta}$ and $\Gamma(\beta)$ is the Euler gamma-function.

For $\hat{q}^{2\alpha}$ the "matrix representation" is: $\quad\langle x|\hat{q}^{2\alpha}|y\rangle=x^{2\alpha}\,\delta^D(x-y)$.

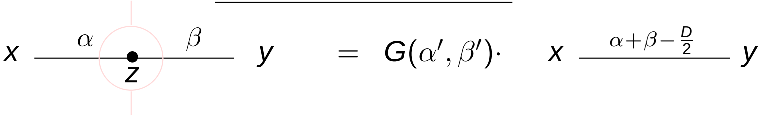
3. Operator formalism

Algebraic relations (a,b,c) which are helpful for analytical calculations of perturbative integrals for multi-loop F.D. ⇒ reconstruction of graphs

a. Group relation. Consider a convolution product of two propagators:

$$\int \frac{d^D z}{(x-z)^{2\alpha}\,(z-y)^{2\beta}} = \frac{G(\alpha',\beta')}{(x-y)^{2(\alpha+\beta-D/2)}}\,, \quad \left(G(\alpha,\beta) = \frac{a(\alpha+\beta)}{a(\alpha)\,a(\beta)} \right),$$

which leads to the reconstruction of graph:



This is the "matrix representation" of the operator relation

$$\boxed{\hat{p}^{-2\alpha'}\,\hat{p}^{-2\beta'} = \hat{p}^{-2(\alpha'+\beta')}.} \quad !!!$$

Proof.

$$\int d^D z \,\langle x|\hat{p}^{-2\alpha'}|z\rangle\,\langle z|\hat{p}^{-2\beta'}|y\rangle = \langle x|\hat{p}^{-2(\alpha'+\beta')}|y\rangle$$

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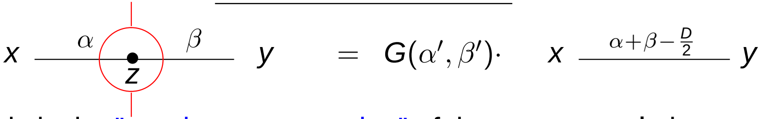

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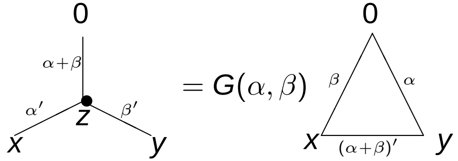
3. Operator formalism

b. Star-triangle relation

The "Method Of Uniqueness" (D.Kazakov, 1983) (Yang-Baxter equation)

$$\int \frac{d^D z}{(x-z)^{2\alpha'}\, z^{2(\alpha+\beta)}\, (z-y)^{2\beta'}} = \frac{G(\alpha,\beta)}{(x)^{2\beta}\, (x-y)^{2(\frac{D}{2}-\alpha-\beta)}\, (y)^{2\alpha}} \, .$$

Reconstruction of graph:



$$\boxed{\alpha' = \frac{D}{2} - \alpha}$$

Operator version:

$$\boxed{\hat{p}^{-2\alpha}\hat{q}^{-2(\alpha+\beta)}\hat{p}^{-2\beta} = \hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)}\hat{q}^{-2\alpha}} \quad !!!$$

Compare with Yang-Baxter equation:

$$S(\alpha)\,\widetilde{S}(\alpha+\beta)\,S(\beta) = \widetilde{S}(\beta)\,S(\alpha+\beta)\,\widetilde{S}(\alpha)$$

3. Operator formalism

Remarks on star-triangle relation:

1. STR is a commutativity condition for the set of operators $H_\alpha = \hat{p}^{2\alpha}\hat{q}^{2\alpha}$:

$$\hat{p}^{2\gamma}\hat{q}^{2\gamma}\,\hat{p}^{2\alpha}\hat{q}^{2\alpha} = \hat{p}^{2\alpha}\hat{q}^{2\alpha}\,\hat{p}^{2\gamma}\hat{q}^{2\gamma} \Rightarrow$$

$$\hat{p}^{2(\gamma-\alpha)}\hat{q}^{2\gamma}\,\hat{p}^{2\alpha} = \hat{q}^{2\alpha}\,\hat{p}^{2\gamma}\hat{q}^{2(\gamma-\alpha)} \Rightarrow \text{STR for } \gamma = \alpha + \beta\ .$$

2. Algebraic proof of the STR. Introduce inversion operator R:

$$R^2 = 1\ , \quad \langle x_i | R = \langle \tfrac{x_i}{x^2} |$$

$$R\hat{q}_i R = \hat{q}_i \,/\, \hat{q}^2\ , \quad R\hat{p}_i R = \hat{q}^2\,\hat{p}_i - 2\,\hat{q}_i\,(\hat{q}\,\hat{p}) =: K_i\ ,$$

$$\boxed{R\,\hat{p}^{2\beta}\,R = \hat{q}^{2(\beta+\frac{D}{2})}\,\hat{p}^{2\beta}\,\hat{q}^{2(\beta-\frac{D}{2})}}\ .$$

Proof.

$$R\,\underset{\overset{\Uparrow}{R^2}}{\hat{p}^{2\alpha}\,\hat{p}^{2\beta}}\,R \;= R\,\hat{p}^{2(\alpha+\beta)}\,R \;\Rightarrow\; \hat{p}^{2\alpha}\hat{q}^{2(\alpha+\beta)}\,\hat{p}^{2\beta} = \hat{q}^{2\beta}\,\hat{p}^{2(\alpha+\beta)}\hat{q}^{2\alpha}$$

3. Operator formalism

3. One can deduce ”local” STR which is related to the α-representation for FD (*R.Kashaev, 1996*)

$$W(x^2|\alpha) = \exp\left(-\frac{x^2}{2\alpha}\right)$$

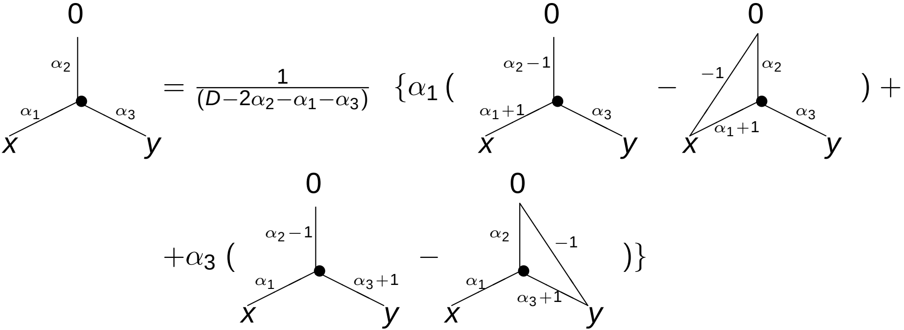
$$\boxed{W(\hat{q}^2|\alpha_1)\,W(\hat{p}^2|\,\tfrac{1}{\alpha_2})\,W(\hat{q}^2|\alpha_3) = W(\hat{p}^2|\,\tfrac{1}{\beta_3})\,W(\hat{q}^2|\beta_2)\,W(\hat{p}^2|\,\tfrac{1}{\beta_1})}$$

where $\alpha_i = \frac{\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3}{\beta_i}$ is a star-triangle transformation for resistances in electric networks

3. Operator formalism

c. Integration by parts rule. (*F. Tkachov, K. Chetyrkin, 1981*)

(reconstruction of graphs)



It can be represented in the operator form:

$$(2\gamma-\alpha-\beta)\,\hat{p}^{2\alpha}\hat{q}^{2\gamma}\hat{p}^{2\beta} = \frac{[\hat{q}^2,\,\hat{p}^{2(\alpha+1)}]}{4(\alpha+1)}\,\hat{q}^{2\gamma}\,\hat{p}^{2\beta} - \hat{p}^{2\alpha}\hat{q}^{2\gamma}\,\frac{[\hat{q}^2,\,\hat{p}^{2(\beta+1)}]}{4(\beta+1)}\;!!$$

where $\alpha = -\alpha_1'$, $\gamma = -\alpha_2$ and $\beta = -\alpha_3'$.

3. Operator formalism

The integration by parts identity

$$(2\gamma-\alpha-\beta)\,\hat{p}^{2\alpha}\hat{q}^{2\gamma}\hat{p}^{2\beta} = \frac{[\hat{q}^2,\,\hat{p}^{2(\alpha+1)}]}{4(\alpha+1)}\,\hat{q}^{2\gamma}\,\hat{p}^{2\beta} - \hat{p}^{2\alpha}\hat{q}^{2\gamma}\,\frac{[\hat{q}^2,\,\hat{p}^{2(\beta+1)}]}{4(\beta+1)}\;,$$

can be proved by using relations for Heisenberg algebra

$$[\hat{q}^2,\,\hat{p}^{2(\alpha+1)}] = 4\,(\alpha+1)\,(H+\alpha)\,\hat{p}^{2\alpha}\;,$$

$$H\,\hat{q}^{2\alpha} = \hat{q}^{2\alpha}\,(H+2\alpha)\;, \quad H\,\hat{p}^{2\alpha} = \hat{p}^{2\alpha}\,(H-2\alpha)\;,$$

where $\boxed{H := \frac{\mathrm{i}}{2}(\hat{p}_i\hat{q}_i + \hat{q}_i\hat{p}_i)}$ is the dilatation operator.

The set of operators $\{\hat{q}^2, \hat{p}^2, H\}$ generates the algebra $sl(2)$.

3. Operator formalism

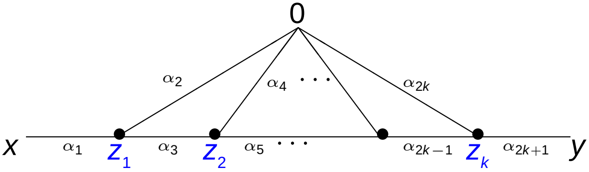
An example of the operator representation for F.D.

Consider an operator:

$$\Psi(\alpha_i) = \hat{p}^{-2\alpha_1'}\,\hat{q}^{-2\alpha_2}\,\hat{p}^{-2\alpha_3'}\,\hat{q}^{-2\alpha_4}\,\hat{p}^{-2\alpha_5'}\cdots\hat{q}^{-2\alpha_{2k}}\,\hat{p}^{-2\alpha_{2k+1}'}\,.$$

This operator is the algebraic version of 3-point function:

$\langle x|\Psi(\alpha_i)|y\rangle \sim$



Indeed,

$$\langle x|\Psi(\alpha_i)|y\rangle = \langle x|\hat{p}^{-2\alpha_1'}\,\underset{\substack{\Uparrow\\ \int d^Dz_1|z_1\rangle\langle z_1|}}{\hat{q}^{-2\alpha_2}}\;\hat{p}^{-2\alpha_3'}\,\underset{\substack{\Uparrow\\ \int d^Dz_2|z_2\rangle\langle z_2|}}{\hat{q}^{-2\alpha_4}}\;\hat{p}^{-2\alpha_5'}\cdots\underset{\substack{\Uparrow\\ \int d^Dz_k|z_k\rangle\langle z_k|}}{\hat{q}^{-2\alpha_{2k}}}\;\hat{p}^{-2\alpha_{2k+1}'}|y\rangle$$

Remark. $\langle x|\Psi(\alpha_i)|x\rangle$ represents the propagator-type diagrams.

3. Operator formalism

The advantage: we change the manipulations with integrals by the manipulations with elements of the algebra generated by $\hat{p}^{2\alpha}$,$\hat{q}^{2\beta}$.

Is it possible to define the trace for this algebra?

$$\mathrm{Tr}\,(\Psi(\alpha_i)) = \int d^D x \, \langle x | \hat{p}^{-2\alpha'_1} \, \hat{q}^{-2\alpha_2} \, \hat{p}^{-2\alpha'_3} \cdots \hat{q}^{-2\alpha_{2k}} \, \hat{p}^{-2\alpha'_{2k+1}} | x \rangle = c(\alpha_i) \int \frac{d^D x}{x^{2\beta}} \, .$$

($\beta = \sum_i \alpha_i$; $c(\alpha_i)$- coeff. function). The dim. reg. procedure requires:

$$\int \frac{d^D x}{x^{2(D/2+\alpha)}} = 0 \quad \forall \alpha \neq 0 \, .$$

The extension of the definition of this integral is (S.Gorishnii, A.Isaev, 1985)

$$\boxed{\int \frac{d^D x}{x^{2(D/2+\alpha)}} = \pi \Omega_D \delta(|\alpha|) \, ,}$$!!!

where $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$, $\alpha = |\alpha| e^{i \arg(\alpha)}$. Then, the cyclic property of "Tr" can be checked. "Tr": propagators $\Rightarrow$ vacuum diagrams.

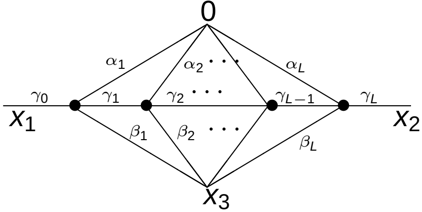
4. Application

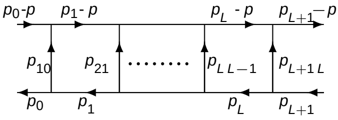
L-loop ladder diagrams for ϕ^3 FT $\Leftrightarrow$ D-dimensional conformal QM

Consider dimensionally and analytically regularized massless integrals

$$D_L(p_0,p_{L+1},p;\vec{\alpha},\vec{\beta},\vec{\gamma}) = \left[\prod_{k=1}^{L}\int \frac{d^D p_k}{p_k^{2\alpha_k}(p_k-p)^{2\beta_k}}\right]\prod_{m=0}^{L}\frac{1}{(p_{m+1}-p_m)^{2\gamma_m}}$$

which correspond to the diagrams ($x_1=p_0$, $x_2=p_{L+1}$, $x_3=p$):





$(p_{mk}=p_m-p_k)$

The diagrams (in config. and moment. spaces) are dual to each other (the boldface vertices correspond to the loops). The operator version is

$$D_L(x_a;\vec{\alpha},\vec{\beta},\vec{\gamma}) \sim \langle x_1|\hat{p}^{-2\gamma_0'}\left(\prod_{k=1}^{L}\hat{q}^{-2\alpha_k}(\hat{q}-x_3)^{-2\beta_k}\hat{p}^{-2\gamma_k'}\right)|x_2\rangle\ .$$

4. Application

For simplicity we put $\boxed{\alpha_i = \alpha, \beta_i = \beta, \gamma_i = \gamma}$ and consider the generating function for D_L:

$$D_g(x_a; \alpha, \beta, \gamma) = \sum_{L=0}^{\infty} g^L D_L(x_a; \alpha, \beta, \gamma) \sim \langle x_1 | \left(\hat{p}^{2\gamma'} - \frac{\bar{g}}{\hat{q}^{2\alpha}(\hat{q} - x_3)^{2\beta}} \right)^{-1} | x_2 \rangle$$

where $\bar{g} = g/a(\gamma')$ is the renormalized coupling constant. For the case $\boxed{\alpha + \beta = 2\gamma'}$, using inversions, etc. we obtain

$$D_g \sim \langle u | \left(\hat{p}^{2\gamma'} - \frac{g_x}{\hat{q}^{2\beta}} \right)^{-1} | v \rangle ,$$

where $g_x = \bar{g}(x_3)^{-2\beta}$, $u_i = \frac{(x_1)_i}{(x_1)^2} - \frac{(x_3)_i}{(x_3)^2}$, $v_i = \frac{(x_2)_i}{(x_2)^2} - \frac{(x_3)_i}{(x_3)^2}$.

The ϕ^3-theory for $D = 4$ is related to $\gamma' = 1 = \beta$ and we obtain the Green's function for conformal QM:

$$D_g \sim \langle u | \left(\hat{p}^2 - \frac{g_x}{\hat{q}^2} \right)^{-1} | v \rangle ,$$

For $D \neq 4$ this GF ⇒ ladder diagrams for $\alpha = \beta = 1, \gamma = \frac{D}{2} - 1$.

4. Application

Our method is based on the identity:

$$\frac{1}{\hat{p}^2 - g/\hat{q}^2} = \sum_{L=0}^{\infty} \left(-\frac{g}{4}\right)^L \left[\hat{q}^{2\alpha} \frac{(H-1)}{(H-1+\alpha)^{L+1}} \frac{1}{\hat{p}^2} \hat{q}^{-2\alpha}\right]_{\alpha^L}$$

where we denote $[\ldots]_{\alpha^L} = \frac{1}{L!} \left(\partial_\alpha^L [\ldots]\right)_{\alpha=0}$. Taking into account

$$\frac{(H-1)}{(H-1+\alpha)^{L+1}} = \frac{(-1)^{L+1}}{L!} \int_0^\infty dt\, t^L\, e^{t\alpha}\, \partial_t \left(e^{t\,(H-1)}\right)$$

and $e^{t(H+\frac{D}{2})}\,|x\rangle = |e^{-t}x\rangle$ the Green's function D_g is written in the form

$$\langle u| \frac{1}{(\hat{p}^2 - g_x/\hat{q}^2)} |v\rangle = \sum_{L=0}^{\infty} \frac{1}{L!} \left(\frac{g_x}{4}\right)^L \Phi_L(u,v)\,,$$

$$\Phi_L(u,v) = -a(1) \int_0^\infty dt\, t^L \left[\left(\frac{u^2}{v^2}\right)^\alpha e^{t\alpha}\right]_{\alpha^L} \partial_t \left(\frac{e^{-t}}{(u-e^{-t}v)^2}\right)^{(\frac{D}{2}-1)}$$

4. Application

For $D = 4 - 2\epsilon$ one can expand $\Phi_L(u, v)$ over small ϵ:

$$\Phi_L(u, v) = \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon} u^{2(1-\epsilon)}} \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \Phi_L^{(k)}(z_1, z_2) \, .$$

where $z_1 + z_2 = 2(uv)/u^2$ and $z_1 z_2 = v^2/u^2$. The coeff. functions $\Phi_L^{(k)}$ are expressed in terms of multiple polylogarithms. The first one is (N.I. Ussyukina and A.I. Davydychev; D.J. Broadhurst; 1993)

$$\Phi_L^{(0)}(z_1, z_2) = \frac{1}{z_1 - z_2} \sum_{f=0}^{L} \frac{(-)^f \, (2L-f)!}{f! \, (L-f)!} \ln^f(z_1 z_2) \left[\mathrm{Li}_{2L-f}(z_1) - \mathrm{Li}_{2L-f}(z_2) \right] .$$

where polylogs are

$$\mathrm{Li}_m(w) = \sum_{n=1}^{\infty} \frac{w^n}{n^m} \, .$$

4. Application

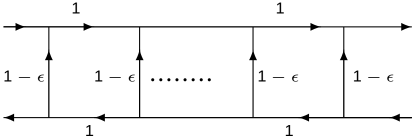
The next coefficient is: $\Phi_L^{(1)}(z_1, z_2) =$

$$= \sum_{n=L}^{2L} \frac{n!\, \ln^{2L-n}(z_1 z_2) \left[\left(n \mathrm{Li}_{n+1}(z_1) - \mathrm{Li}_{n,1}(z_1, 1) - \mathrm{Li}_{n,1}(z_1, \frac{z_2}{z_1}) \right) - (z_1 \leftrightarrow z_2) \right]}{(-1)^n\, (2L-n)!\, (n-L)! \;\; (z_1 - z_2)},$$

where multiple polylogarithms are

$$\mathrm{Li}_{m_0, m_1, \ldots, m_r}(w_0, w_1, \ldots, w_r) = \sum_{n_0 > n_1 > \cdots > n_r > 0} \frac{w_0^{n_0} w_1^{n_1} \cdots w_r^{n_r}}{n_0^{m_0} n_1^{m_1} \ldots n_r^{m_r}}\,.$$

The function $\Phi_L^{(1)}(z_1, z_2)$ gives the first term in the expansion over ϵ of the L-loop ladder diagram (with special indices on the lines)



5. Application to Lipatov's model

Lipatov's model is described by the Hamiltonian $H = \sum_{i=1}^{n} H_{ii+1}$, where

$$H_{ik} = \hat{p}_i \ln(\rho_{ik}) \hat{p}_i^{-1} + \hat{p}_k \ln(\rho_{ik}) \hat{p}_k^{-1} + \ln(\hat{p}_i \hat{p}_k) - 2\psi(1) = \tag{1}$$

$$= 2 \ln(\rho_{ik}) + \rho_{ik} \ln(\hat{p}_i \hat{p}_k)\, \rho_{ik}^{-1} - 2\psi(1)\,. \tag{2}$$

Here $\psi(1)$ - constant, $\rho_{ik} = q_i - q_k$, q_i - coordinates, $\hat{p}_i = -i \frac{\partial}{\partial q_i}$ - momenta.

Expression (2) appears in the expansion over ϵ of the operator

$$R_{ik}(\epsilon) := \rho_{ik}^{1+\epsilon} (\hat{p}_i \hat{p}_k)^{\epsilon} \rho_{ik}^{-1+\epsilon} = 1 + \epsilon \left(2 \ln(\rho_{ik}) + \rho_{ik} \ln(\hat{p}_i \hat{p}_k)\, \rho_{ik}^{-1}\right) + \epsilon^2 \dots\,.$$

One-dimensional analog of the operator "star-triangle" identity:

$$\rho_{ik}^{\alpha}\, \hat{p}_i^{\alpha+\beta}\, \rho_{ik}^{\beta} = \hat{p}_i^{\beta}\, \rho_{ik}^{\alpha+\beta}\, \hat{p}_i^{\alpha} \;\Leftrightarrow\; \rho_{ki}^{\alpha}\, \hat{p}_i^{\alpha+\beta}\, \rho_{ki}^{\beta} = \hat{p}_i^{\beta}\, \rho_{ki}^{\alpha+\beta}\, \hat{p}_i^{\alpha}\,.$$

Then, we have

$$\rho_{ik}^{1+\epsilon} (\hat{p}_i \hat{p}_k)^{\epsilon} \rho_{ik}^{-1+\epsilon} = \rho_{ik}^{1+\epsilon}\, \hat{p}_i^{\epsilon}\, \rho_{ik}^{-1}\, \rho_{ik}^{1}\, \hat{p}_k^{\epsilon}\, \rho_{ik}^{-1+\epsilon} = \hat{p}_i^{-1}\, \rho_{ik}^{\epsilon}\, \hat{p}_i^{1+\epsilon}\, \hat{p}_k^{-1+\epsilon}\, \rho_{ik}^{\epsilon}\, \hat{p}_k^{1} =$$

$$= 1 + \epsilon \left(\hat{p}_i^{-1} \ln(\rho_{ik})\, \hat{p}_i + \hat{p}_k^{-1} \ln(\rho_{ik})\, \hat{p}_k + \ln(\hat{p}_i \hat{p}_k)\right) + \epsilon^2 \dots\,,$$

and this proves the equivalence of the expressions (1) and (2).

5. Application to Lipatov's model

The operator $R_{ik}(\epsilon) := \rho_{ik}^{1+\epsilon}(\hat{p}_i\hat{p}_k)^{\epsilon}\rho_{ik}^{-1+\epsilon}$ satisfies the Yang-Baxter equation

$$R_{i\,i+1}(\epsilon)\,R_{i+1\,i+2}(\epsilon+\epsilon')\,R_{i\,i+1}(\epsilon') = R_{i+1\,i+2}(\epsilon')\,R_{i\,i+1}(\epsilon+\epsilon')\,R_{i+1\,i+2}(\epsilon)\,.$$

Then the complete holomorphic Hamiltonian $H = \sum_{i=1}^{n} H_{ii+1}$ appears in the expansion over ϵ of the monodromy matrix

$$T_{(1,2\ldots,n+1)}(\epsilon) = R_{1\,2}(\epsilon)\,R_{2\,3}(\epsilon)\,R_{3\,4}(\epsilon)\cdots R_{n\,n+1}(\epsilon)\ .$$

Recent results of (S.E. Derkachov and A.N.Manashov, "R-Matrix and Baxter Q-Operators for the Noncompact $SL(N,\mathbf{C})$ Invariant Spin Chain", nlin.SI/0612003, 2006) generalize the constructions presented above.

Summary

- Applications of the coefficients $\Phi_L(u,v)$ for the avaluations of 4-point functions in $N=4$ SYM theory.
- Lipatov's integrable model – describes high energy scattering of hadrons in QCD.
- Generalizations to massive case and to supersymmetric case. In massive case it is tempting to calculate the Green's function

$$\langle u| \frac{1}{(\hat{p}^2 - g/\hat{q}^2 + m^2)} |v\rangle = \sum_{L=0}^{\infty} g^L \, \Phi_L(u,v;m^2) \; ,$$

- It seems that the approach is not universal even for massless FDs. We should add something new.

For Further Reading I

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Multi-Loop Feynman Integrals and Conformal Quantum Mechanics