Nonperturbative quantum relativistic effects in the confinement mechanism for particles in a deep potential well.

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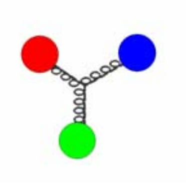
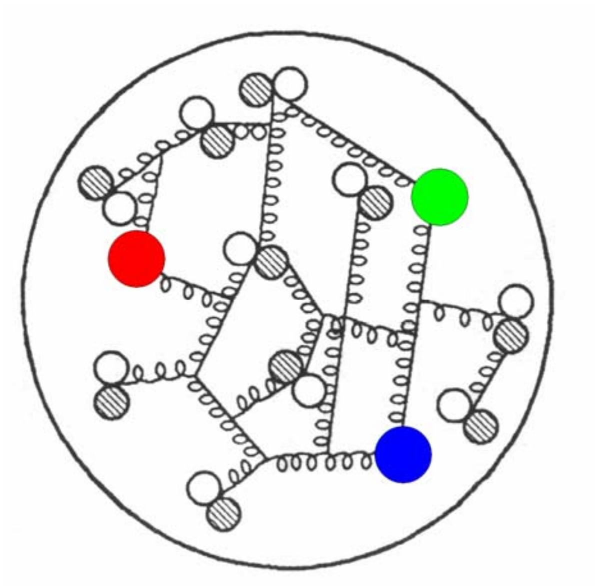


$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\gamma_\mu D^\mu - m)q - \frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$$



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Quasipotential approach for description of relativistic bound states:

1. A. A. Logunov, A. N. Tavkhelidze, Nuovo Cimento **29** (1963) 380;
2. V. G. Kadyshevsky, R. M. Mir-Kasimov, N. B. Skachkov, Part. and Nucl. **2** (1972) 635;
3. K. A. Sveshnikov, Phys. Lett. A **136** (1989), 1;
4. V. A. Matveev, V. I. Savrin, A. N. Sisakian, A. N. Tavkhelidze, TMF **132** (2002), 267.

Main difficulty — finite pure imaginary shift operators in equations:

$$\hat{A} = \exp(\pm i\lambda \frac{d}{d\xi})$$

$$\hat{A}\psi(\xi) = \psi(\xi \pm i\lambda)$$

The quasipotential equation for a scalar particle in 1+1D:

$$\frac{1}{\lambda^2}[\phi(\xi - i\lambda) + \phi(\xi + i\lambda) - 2\phi(\xi)] + V(\xi)\phi(\xi) = E\phi(\xi)$$

$$V(\xi) = V_0^2\theta(|\xi| - \xi_0)$$

$$E = \frac{4}{\lambda^2}\sinh^2\frac{\lambda\omega}{2}$$

If we restore dimensions:

$$\xi = Mx; \qquad \phi(\xi \pm i\lambda) \longrightarrow \phi(x \pm i\lambda/M)$$

Physical demands of wavefunction:

1. Square-integrable on real axis.
2. $\phi(\xi)$ is analytic function in the strip $|\mathrm{Im}\ \xi| < \lambda$

Kinetic energy:

$$K[\psi] = -\frac{1}{\lambda^2} \int d\xi (\psi(\xi + i\lambda) - \psi(\xi))^* (\psi(\xi - i\lambda) - \psi(\xi))$$

General form of the solution:

$$\phi_{in}(\xi) = 2 \sum_{n=-\infty}^{+\infty} A_n \cos(\omega_n \xi), \qquad |Re\xi| < \xi_0$$

$$\phi_{out}(\xi) = \phi_{out}(-\xi) = \sum_{s=0}^{+\infty} (B_s e^{-\kappa_s \xi} + \bar{B}_s e^{-\bar{\kappa}_s \xi}); \qquad Re\xi > \xi_0$$

where:

$$\bar{E} = {V_0}^2 - E = \frac{4}{\lambda^2} \sin^2 \frac{\lambda\kappa}{2}; \quad \bar{\kappa} = \frac{2\pi}{\lambda} - \kappa$$

$$i\omega_n = i\omega + \frac{2\pi n}{\lambda}; \quad \kappa_s = \kappa + \frac{2\pi s}{\lambda}; \quad \bar{\kappa}_s = \bar{\kappa} + \frac{2\pi s}{\lambda}$$

Integral form of the equation:

$$\phi(\xi) = {V_0}^2 \int\limits_{-\xi_0}^{\xi_0} d\xi' G(\xi - \xi')\phi(\xi')$$

where *G* is Green function for operator:

$$(\Lambda\phi)(\xi) = \frac{1}{\lambda^2}(\phi(\xi - i\lambda) + \phi(\xi + i\lambda) - 2\phi(\xi)) + |\bar{E}|\phi(\xi)$$

Infinite set of equations for coefficients A_n:

$$\sum_{n=-\infty}^{+\infty} A_n \left(\frac{e^{i\omega_n\xi_0}}{\kappa_s - i\omega_n} + \frac{e^{-i\omega_n\xi_0}}{i\omega_n + \kappa_s} \right) = 0; \qquad s = 0 \ldots + \infty$$

where:

$$i\omega_n = i\omega + \frac{2\pi n}{\lambda}; \quad \kappa_s = \kappa + \frac{2\pi s}{\lambda}; \quad \bar{\kappa}_s = \bar{\kappa} + \frac{2\pi s}{\lambda}$$

The condition for "quasi-exact" solution:

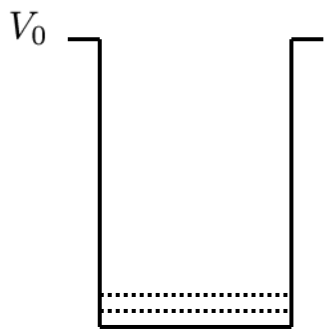
$$e^{-\frac{2\pi\xi_0}{\lambda}} \ll 1$$

(in hadron physics $\xi_0 \approx 2; \quad \lambda \approx (??)\frac{1}{N_c} = \frac{1}{3}$)

Spectral equation for bosons:

$$e^{2i\omega\xi_0} = \frac{\Gamma(-\frac{i\omega\lambda}{\pi})}{\Gamma(\frac{i\omega\lambda}{\pi})} \frac{\Gamma(\frac{\lambda}{2\pi}(i\omega+\kappa))\Gamma(\frac{\lambda}{2\pi}(i\omega+\bar{\kappa}))}{\Gamma(\frac{\lambda}{2\pi}(\kappa-i\omega))\Gamma(\frac{\lambda}{2\pi}(\bar{\kappa}-i\omega))}$$

For low-lying levels in sufficiently deep well:



$$\kappa = \frac{\pi}{\lambda} + i\alpha, \quad \omega \ll \alpha, \quad \frac{\lambda\alpha}{2\pi} \gg 1, \quad {V_0}^2 \gg \frac{4}{\lambda^2}$$

this equation can be strongly simplified:

$$e^{2i\omega(\xi_0-\Delta\xi_0)} = \pm 1, \quad \Delta\xi_0 = \frac{\lambda}{\pi}\ln\frac{\ln\lambda V_0}{2\pi}$$

Region of parameters, where this analysis is valid:

$$1 \ll \left(\frac{\lambda V_0}{2}\right)^2 \ll e^{2\pi e^{\frac{\pi\xi_0}{\lambda}}}$$ ($\sim 1000^{1000}$ for this problem)

The finite-difference analogue of squared Dirac equation in 1+1D:

$$\frac{1}{\lambda^2}[\chi(\xi - i\lambda) + \chi(\xi + i\lambda) - 2\chi(\xi)] + [V(\xi) + i\gamma^1\rho(\xi)]\chi(\xi) = E\chi(\xi)$$

$$\rho(\xi) = V_0[\delta(\xi - \xi_0) - \delta(\xi + \xi_0)]$$

$$V(\xi) = V_0^2\theta(|\xi| - \xi_0)$$

$$E = \frac{4}{\lambda^2}\sinh^2\frac{\lambda\omega}{2}$$

$\chi(\xi)$ is analytic function in the strip $|\mathrm{Im}\ \xi| < \lambda$

Equations for spinor components:

$$\begin{cases} \frac{1}{\lambda^2}[a(\xi - i\lambda) + a(\xi + i\lambda) - 2a(\xi)] + \bar{E}a(\xi) = V_0{}^2\theta(\xi_0 - |\xi|)a(\xi) + i\rho(\xi)b(\xi) \\ \frac{1}{\lambda^2}[b(\xi - i\lambda) + b(\xi + i\lambda) - 2b(\xi)] + \bar{E}b(\xi) = V_0{}^2\theta(\xi_0 - |\xi|)b(\xi) - i\rho(\xi)a(\xi) \end{cases}$$

General form of the solution inside the well:

$$a_{in} = 2 \sum_{n=-\infty}^{+\infty} A_n \cos(\omega_n \xi), \qquad |\mathrm{Re}\ \xi| < \xi_0$$

$$b_{in} = 2i \sum_{n=-\infty}^{+\infty} B_n \sin(\omega_n \xi), \qquad |\mathrm{Re}\ \xi| < \xi_0$$

and outside the well:

$$a_{out}(\xi) = a_{out}(-\xi) = \sum_{s=0}^{+\infty} (C_s e^{-\kappa_s \xi} + \bar{C}_s e^{-\bar{\kappa}_s \xi}), \qquad \mathrm{Re}\ \xi > \xi_0$$

$$b_{out}(\xi) = -b_{out}(-\xi) = \sum_{s=0}^{+\infty} (D_s e^{-\kappa_s \xi} + \bar{D}_s e^{-\bar{\kappa}_s \xi}), \qquad \mathrm{Re}\ \xi > \xi_0$$

where:

$$\bar{E} = {V_0}^2 - E = \frac{4}{\lambda^2} \sin^2 \frac{\lambda \kappa}{2}; \quad \bar{\kappa} = \frac{2\pi}{\lambda} - \kappa$$

$$i\omega_n = i\omega + \frac{2\pi n}{\lambda}; \quad \kappa_s = \kappa + \frac{2\pi s}{\lambda}; \quad \bar{\kappa}_s = \bar{\kappa} + \frac{2\pi s}{\lambda}$$

Infinite set of equations for coefficients A_n:

$$\begin{cases} \sum_{n=-\infty}^{+\infty} A_n \left(\frac{e^{i\omega_n \xi_0}}{\kappa_s - i\omega_n} + \frac{e^{-i\omega_n \xi_0}}{i\omega_n + \kappa_s} \right) - \frac{ib(\xi_0)}{V_0} = 0 \\ \sum_{n=-\infty}^{+\infty} A_n \left(\frac{e^{i\omega_n \xi_0}}{\bar{\kappa}_s - i\omega_n} + \frac{e^{-i\omega_n \xi_0}}{i\omega_n + \bar{\kappa}_s} \right) - \frac{ib(\xi_0)}{V_0} = 0 \end{cases} ; \qquad s = 0 \ldots \infty$$

Under the same assumption:

$$e^{-\frac{2\pi \xi_0}{\lambda}} \ll 1$$

we derive spectral equation for fermions:

$$e^{2i\omega\xi_0} = S(\gamma) \frac{\Gamma(-\frac{i\omega\lambda}{\pi})}{\Gamma(\frac{i\omega\lambda}{\pi})} \frac{\Gamma(\frac{\lambda}{2\pi}(i\omega + \kappa))\Gamma(\frac{\lambda}{2\pi}(i\omega + \bar{\kappa}))}{\Gamma(\frac{\lambda}{2\pi}(\kappa - i\omega))\Gamma(\frac{\lambda}{2\pi}(\bar{\kappa} - i\omega))}$$

$$S(\gamma) = -\frac{i\omega - \gamma(\omega, \lambda, V_0)}{i\omega + \gamma(\omega, \lambda, V_0)}$$

and simplify it for low-lying levels in deep well:

$$e^{2i\omega(\xi_0 - \Delta_1 \xi_0)} = \pm 1, \quad \Delta_1 \xi_0 - \Delta\xi_0 \approx \lambda \frac{\ln^2(\lambda V_0)}{(V_0 \lambda)^2}$$

Comparison of boson and fermion spectrum in case of differential and finite-difference equations.

Differential Schrödinger equation:

$$\omega_n = \frac{\pi + \pi n}{\xi_0}$$

Differential Dirac equation:

$$\omega_n = \frac{\frac{\pi}{2} + \pi n}{\xi_0}$$

Finite-difference equations.

Spectral equation for bosons:

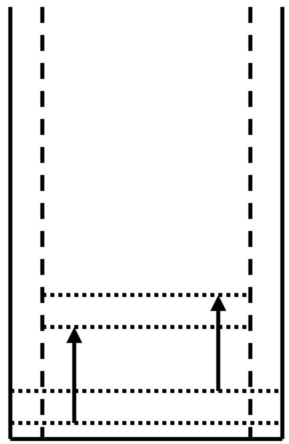
$$e^{2i\omega(\xi_0 - \Delta\xi_0)} = \pm 1, \quad \Delta\xi_0 = \frac{\lambda}{\pi}\ln\frac{\ln\lambda V_0}{2\pi}$$

and for fermions:

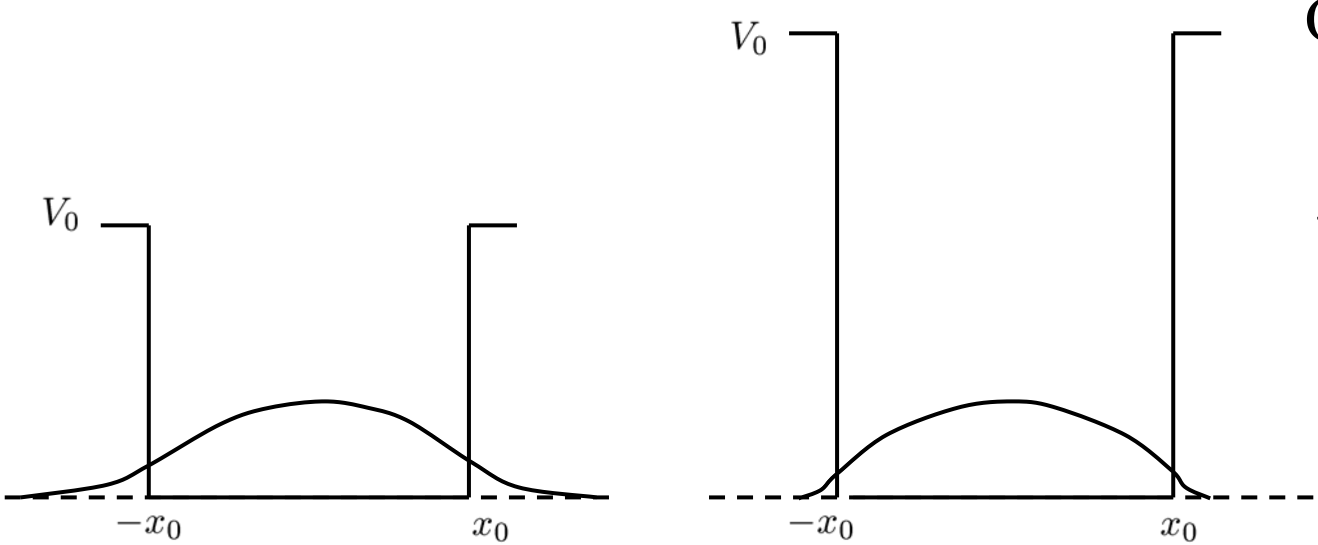
$$e^{2i\omega(\xi_0 - \Delta_1\xi_0)} = \pm 1, \quad \Delta_1\xi_0 - \Delta\xi_0 \approx \lambda\frac{\ln^2(\lambda V_0)}{(V_0\lambda)^2}$$

Two main effects:

1. The energy levels of fermions coincide with boson one the closer the larger is V_0.
2. Effective contraction of the well and increase of the energy levels in comparison with Scrödinger levels in infinitely deep well.



Qualitative behavior of the wavefunctions.
Differential equations:



Outside the well:

$$\phi \sim \exp\left(-K|x|\right)$$

$$K \sim \sqrt{V_0 - E}$$

Boundary conditions for particles confined in infinitely deep well:
for bosons:

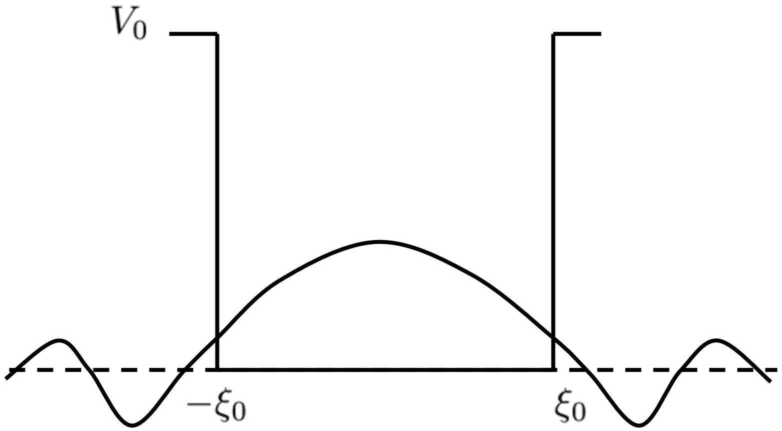
$$\phi(\pm x_0, t) = 0$$

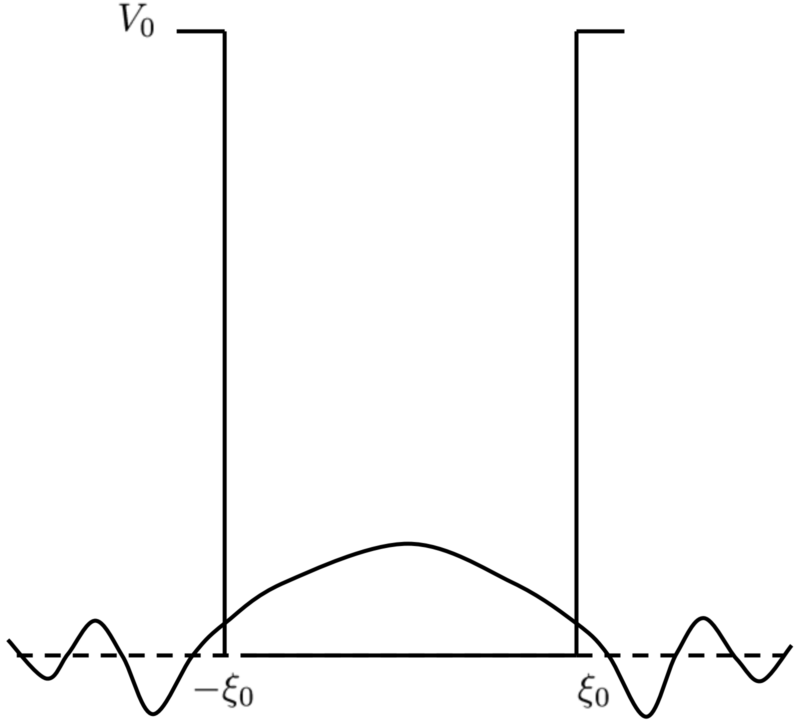
for fermions:

$$(i\gamma^1 \pm 1)\psi(\pm x_0, t) = 0$$

Qualitative behavior of the wavefunctions.

Finite-difference equations:





Outside the well on real axis:

$$\psi = \sum_{s=0}^{+\infty} (C_s e^{-\kappa_s|\xi|} + \bar{C}_s e^{-\bar{\kappa}_s|\xi|}) \approx \exp\left(-\frac{\pi + i\alpha}{\lambda}|\xi|\right)$$

SUMMARY

Method for solution of finite-difference equations was developed.

Main properties of solutions of finite-difference analogues of Schrödinger and Dirac equations in 1+1D:

1. Energy levels of fermions coincide with boson one the closer the larger is V_0.
2. Effective contraction of the well and increase of the energy levels in comparison with Schrödinger levels in infinitely deep well.
3. Wavefunctions always penetrates into forbidden zone - decreasing rate outside well does not depend on the depth of the energy level in it.

This facts on qualitative level correlate with experimental properties of hadrons and mesons.