Scalar and spinor particles with low binding energy in a strong stationary magnetic field in two and three dimensions

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We discuss

Equations for the bound one-active electron states based on analytic solutions of the Schrödinger and Pauli equations for an uniform magnetic field and single attractive $\delta(r)$ potential

Electron ground states in a magnetic field differ essentially from the analogous state of spin-0 particles

We show

1. The binding energy equations can be obtained without using the boundary conditions in the δ potential model

We calculate the energy level displacement analytically and demonstrate nonlinear dependence on the field intensity

2. The magnetic field indeed plays a stabilizing role in considered systems in the case of weak intensity, but a strong magnetic field acts in the opposite way

These properties may be important for real quantum mechanical fermionic systems in two and three dimensions

I. FORMULATION OF THE PROBLEM

Our main purpose is to derive equations for the binding energy of fermion in a field containing an attractive singular potential and a stationary external magnetic field in the two- and three-dimensional cases.

δ-potential approximation:

- Multielectronic atom field
- Negative ion field
- Field of nuclear forces

Energy level displacements can be seen for a particle in a δ-potential and a magnetic field

2+1 dimensions ⟹ axially symmetric quantum systems of electrically charged fermions: the quantum Hall effect
high temperature superconductivity
various film defects

3+1 dimensions ⟹ real quantum mechanical fermionic systems:
multielectronic atoms, ions

QM method: the expansion of the unknown wave function in a series at the eigenfunctions obtained in the pure magnetic field (differs in principle from the traditional derivation of wave functions using the boundary condition typical for the δ-potential)

II. A scalar particle in an attractive potential in the presence of a uniform magnetic field

Consider a charge in a uniform magnetic field **B** specified as

$$\mathrm{B} = (0, 0, \mathrm{B}) = \nabla \times A, \qquad A = (-yB, 0, 0)$$

The Hamiltonian is

$$\hat{H}_{Sh} = \frac{1}{2m}\left(-i\hbar \frac{\partial}{\partial x} + \frac{eB}{c} y \right)^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}$$

e and *m* are the charge and the mass of the particle.

The particle wave function in this field has the form

$$\Psi_{np_x p_z}(t, \vec{r}) = e^{-E_n t/\hbar} e^{i(xp_x + zp_z)/\hbar} U_n(Y)$$

where

$$E_n = \hbar\omega\left(n + \frac{1}{2} \right) + \frac{p_z^2}{2m}$$

is the electron energy spectrum, $\omega = |eB| / mc.$

The functions $U_n(Y)$ are expressed in terms of the Hermit polynomials, the integer *n=0,1,* 2,.. indicates the Landau level number.

Simple solvable model $\Longrightarrow$ Scalar particle in the three dimensional case in a single attractive δ-potential in the presence of a uniform magnetic field

The corresponding Schrödinger equation is

$$\frac{1}{2m}\left[\left(-i\hbar\frac{\partial}{\partial x}+\frac{eB}{c}y\right)^2-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}-\lambda\hbar^2\delta(\vec{r})\right]\Psi_{E'}(\vec{r})=E'\Psi_{E'}(\vec{r}).$$

The solution has the form

$$\Psi_{E'}(\vec{r})=\sum_{n,p_x p_z}C_{E'np_x p_z}\psi_{n,p_x p_z}(\vec{r})=\sum_{n=0}^{\infty}\int dp_x dp_z C_{E'n,p_x p_z}\psi_{np_x p_z}(\vec{r}),$$

where $\psi_{np_x p_z}(\vec{r})$ is the spatial part of wave function, *E'*= - |*E'*| is the required negative energy.

The coefficients $C_{E'np_x p_z}$ can be easily calculated and we obtain the equation

$$1=N\pi\sqrt{2m\hbar\omega}\sum_{n=0}^{\infty}\frac{1}{(n+A)^{1/2}},\qquad \text{where}\qquad A=\frac{1}{2}-\frac{E'}{\hbar\omega}+\frac{p_z^2}{2m\hbar\omega}$$

This equation implicitly defines the energy of a bound localized electron state in the magnetic field and can be analytically reduced to a simpler form using representation

$$\frac{1}{(n+A+i\varepsilon)^{1/2}}=\frac{e^{-i\pi/4}}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{i(n+A+i\varepsilon)t}}{t^{1/2}}dt$$

We obtain the real expression

$$-1 = N_1 \frac{\sqrt{\hbar\omega}}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-Et/\hbar\omega}}{t^{1/2}\sinh(t/2)} dt, \qquad E = |E'| \geq 0.$$

If we eliminate the magnetic field, the equation takes the form

$$-1 = N_1 \frac{\sqrt{\hbar}}{\sqrt{\pi}} \int_0^\infty \frac{e^{-Et/\hbar}}{t^{3/2}} dt,$$ where $E_0 = |E_0'|$

is the absolute value of the binding energy without the action of the external field

Subtracting equations and removing the integral divergences, we obtain

$$\boxed{\sqrt{E} - \sqrt{E_0} = \frac{\sqrt{E}}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-x}}{x^{3/2}} \left(\frac{ax}{\sinh(ax)} - 1 \right) dx,} \qquad a = \frac{\hbar\omega}{2E}$$

This equation is consistent with the analogous equation obtained by well-known method using boundary conditions of wave functions in the δ-potential model

Yu.N.Demkov, G.F.Drukarev (1965)
V.S. Popov, B.M. Karnakov, V.D. Mur (1998)
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V.N.Rodionov, A.M.Mandel, E.V.Arbuzova (2005)

Expanding the integrand function **in the weak field limit** $\hbar\omega \langle\langle 2E_0$, we get

$$E = E_0\left(1 - \frac{1}{48}\frac{\hbar^2\omega^2}{E_0^2} + \frac{1}{576}\frac{\hbar^4\omega^4}{E_0^4}\right)$$

The restriction to the binding-energy spectrum of the scalar particle in the magnetic field is

$$E' < \frac{\hbar\omega}{2}$$

The explicit equation for the bound-state energy **in the strong-field limit** $\hbar\omega > 2E_0$

$$E' = \hbar\omega\left(0.205 - 0.452\sqrt{\frac{E_0}{\hbar\omega}} - 0.367\frac{E_0}{\hbar\omega}\right)$$

In the super-strong magnetic field

$$E' = 0.205\hbar\omega$$

The upper limit for the binding energy of a scalar particle

This limiting value is independent of the particle energy In the absence of the field and is completely determined by the magnetic field intensity

The two-dimensional model

The analogue of the equation ✦

$$\ln\frac{E}{E_0}=\int_0^\infty \frac{e^{-x}}{x}\left(\frac{ax}{\sinh(ax)}-1\right)dx$$

In the weak field limit

$$\longrightarrow\quad E=E_0\left(1-\frac{\hbar^2\omega^2}{24E_0^2}\right)$$

In the strong-field limit $\hbar\omega > 2E_0$ the expression which explicitly determines the bound state energy is

$$\frac{E'}{\hbar\omega}=\frac{1}{2}-\frac{6\left[C-\ln(\hbar\omega/E_0)\right]+\sqrt{24\pi^2+36\left[C-\ln(\hbar\omega/E_0)\right]^2}}{2\pi^2}$$

C=0.577.. is the Euler constant

For the **super-strong magnetic field** $\ln(\hbar\omega/E_0)>>1$ we obtain

$$\frac{E'}{\hbar\omega}=\frac{1}{2}-\frac{1}{\ln(\hbar\omega/E_0)}-\frac{C}{\ln^2(\hbar\omega/E_0)}+\frac{\pi^2/6-C}{\ln^3(\hbar\omega/E_0)}+O\left[\ln^{-4}(\hbar\omega/E_0)\right]$$

This expansion is correct for the binding energy $E'\le\hbar\omega/2$

Difference from three-dimensional case: for super-strong magnetic field the upper limit of the shifted binding-energy level in the considered model tends directly to the boundary of the continuous spectrum

III. A spin particle in an attractive potential in the presence of a uniform magnetic field

To study spin effects in magnetic fields we can use the same approach based on exact solutions of the Pauli equation with **the Hamiltonian**

$$\hat{H} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} + \frac{eB}{c}y\right)^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \mu\sigma_3 B,$$

Interaction of the electron spin magnetic moment with the magnetic filed

where $\mu = |e|\hbar / 2mc$ is the Bohr magneton,

σ_3 is the ***z*** component of the Pauli matrices

The electron wave functions in the field

$$\psi_{np_xp_zs}(t,\vec{r}) = \frac{1}{2}\psi_{np_xp_z}(t,\vec{r})\begin{pmatrix}1+s\\1-s\end{pmatrix},$$

The solution of Schrödinger equation
In the magnetic field

The electron energy spectrum

$$E_{ns} = \hbar\omega\left(n+\frac{1}{2}\right) + \frac{p_z^2}{2m} + s\hbar\omega\frac{1}{2}$$

$s = \pm 1$ is the conserved spin quantum number

Continuous spectrum boundaries

$E' \geq \hbar\omega/2$	$E' \geq \hbar\omega$	$E' \geq 0$
Scalar particle	**Spin along the magnetic field** ***s*=+1**	**Spin against the magnetic field** ***s*=-1**

The energy equation in the three dimensional case

$$\sqrt{E}-\sqrt{E_0}=\frac{\sqrt{E}}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{t^{3/2}}\left(\frac{ate^{-sat}}{\sinh(at)}-1\right)dt$$

The expansion in **the weak field limit**

$$\frac{E}{E_0}=1-s\frac{\hbar\omega}{2E_0}-\frac{1}{48}\frac{\hbar^2\omega^2}{E_0^2}$$

The energy level

$$E_0=|E'_0|$$

In the δ-potential without perturbation

shifted under the magnetic field action

$$+\frac{\hbar\omega}{2},\quad s=1$$

$$-\frac{\hbar\omega}{2},\quad s=-1$$

The depth of the arrangement of energy levels with respect to the continuous spectrum boundaries is the same in these two cases and in the case of spin-0 particles

In **the strong field limit** $\hbar\omega > E_0$ for different spin values $s=0,+1,-1$

$$E'=\hbar\omega\left(0.205+\frac{s}{2}-0.452\sqrt{\frac{E_0}{\hbar\omega}}-0.367\frac{E_0}{\hbar\omega}\right),$$

The dependence of energy level shifts on the particle spin does not disappear in the strong field limit: The continuous spectrum boundaries are shifted for *s=0, s=1.*

The displacements of the binding energy levels are at the same distances from the continuous spectrum boundaries in all cases

The two-dimensional model

According to our approach
$$\ln\frac{E}{E_0}=\int_0^\infty \frac{e^{-x}}{x}\left(\frac{axe^{-sax}}{\sinh(ax)}-1\right)dx,$$

$s = \pm 1$ represents the particle spin direction

In **the weak field limit**
$$\frac{E}{E_0}=1-\frac{s}{2}\frac{\hbar\omega}{E_0}-\frac{1}{24}\left(\frac{\hbar\omega}{E_0}\right)^2$$

The strong field limit $\ln(\hbar\omega/E_0)\rangle\rangle 1$ gives

For *s*=-1
$$E'=-\hbar\omega\left(\frac{1}{\ln(\hbar\omega/E_0)}+\frac{C}{[\ln(\hbar\omega/E_0)]^2}\right)$$

For the opposite spin orientation s=1 we obtain
$$E'=\hbar\omega\left(1-\frac{1}{\ln(\hbar\omega/E_0)}-\frac{C}{[\ln(\hbar\omega/E_0)]^2}\right)$$

The main difference from the three-dimensional case: the perturbative binding-energy levels converge to the continuous spectrum boundaries in a super strong magnetic field

IV. Conclusions

The effect of a magnetic field on localized electron states leads to equations for The binding energy of spin-0 and spin-1/2 particles

In the weak field limit $\hbar\omega \langle\langle E_0$ **the energy displacements of scalar and spinor particles are described by the expressions**

$$s=0: \qquad E = E_0 - \frac{\hbar^2\omega^2}{24\delta E_0},$$

$$s=1: \qquad E = E_0 - \frac{\hbar\omega}{2} - \frac{\hbar^2\omega^2}{24\delta E_0},$$

$$s=-1: \qquad E = E_0 + \frac{\hbar\omega}{2} - \frac{\hbar^2\omega^2}{24\delta E_0},$$

$\delta = 2$ Three-dimensional case

$\delta = 1$ Two-dimensional case

The dependence on the particle spin does not disappear in the strong field limit $\hbar\omega \rangle\rangle E_0$

Three-dimensional case: the perturbative energy levels approach specific spectral values determined by the magnetic field intensity

The displacements of the binding energy levels are at identical distances from the continuous spectrum boundaries

The strong field limit $\hbar\omega \gg E_0$

Three-dimensional case

$$s=0: \quad E' = 0.205\hbar\omega - 0.452\sqrt{\frac{E_0}{\hbar\omega}} - 0.367\frac{E_0}{\hbar\omega},$$

$$s=1: \quad E' = 0.705\hbar\omega - 0.452\sqrt{\frac{E_0}{\hbar\omega}} - 0.367\frac{E_0}{\hbar\omega},$$

The value of the binding energy level is positive

$$s=-1: \quad E' = -0.295\hbar\omega - 0.452\sqrt{\frac{E_0}{\hbar\omega}} - 0.367\frac{E_0}{\hbar\omega}.$$

Two-dimensional case

$$s=0: \quad E' = \frac{\hbar\omega}{2} - \frac{\hbar\omega}{\ln(\hbar\omega/E_0)} - \frac{C\hbar\omega}{\ln^2(\hbar\omega/E_0)},$$

$$s=1: \quad E' = \hbar\omega - \frac{\hbar\omega}{\ln(\hbar\omega/E_0)} - \frac{C\hbar\omega}{\ln^2(\hbar\omega/E_0)},$$

$$s=-1: \quad E' = -\frac{\hbar\omega}{\ln(\hbar\omega/E_0)} - \frac{C\hbar\omega}{\ln^2(\hbar\omega/E_0)}.$$

The energy levels in the basic terms are independent of the particle energy in the absence of the magnetic field

For super strong magnetic field

$$\ln(\hbar\omega/E_0) \gg 1$$

the binding energy levels approach the continuous spectrum boundaries for all spin values

Summary

1. The energy levels of a polarized electron under the action of a weak magnetic field for different particle spin values are shifted similarly in the three-dimensional and two-dimensional models.

   There are the line displacements as the levels themselves for *s=1* and *s=-1* and analogous shifts of the continuous spectrum boundaries for *s=1.*

   There is the same picture in the case of a spinless particle with the line shift of the continuous spectrum boundary.

   In case of weak intensity a magnetic field indeed plays a stabilizing role in the considered systems because the depth of the perturbative binding energy levels from the continuous spectrum boundaries are shifted downward under the field action independently of the particle spin.

2. Our results show a nonlinear dependence on the field intensity in the strong-field limit.

   The continuous spectrum boundaries in the cases *s=0* and *s=1,* as before, have a linear dependence on the field in this limit.

   In super strong magnetic fields, the binding energy levels can approach the continuous spectrum boundaries.

3. **The distinctions can be formulated as follows:**

In the three-dimensional model, there is a fixed depth of the energy levels from the continuous spectrum boundaries that is the same for all spin values.

In the two-dimensional model, the energy levels in a super strong magnetic field tends asymptotically to the continuous spectrum boundaries.

In the both cases, the system instability increases in strong magnetic fields.

! **This conclusion disproves the opinion that a magnetic field always plays a stabilizing role in systems of bound particles.**

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