QUARK MIXING IN THE STANDARD MODEL AND THE SPACE ROTATIONS

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Main topics

- Generic notes on the Cabibbo-Kobayashi-Maskawa (CKM) matrix in the Standard Model.
- The standard form of the CKM matrix with conserved CP and the rotation matrix (notes only).
- The exponential parameterisation of the mixing matrix and the rotation matrix.
- CP-violation in CKM, exponential parameterisation and the rotation direction vector.
- Generation of new unitary parameterisations of CKM matrix with distinguished CP violating term.

Quark Mixing and Space Rotations

Generic notes on the Quark Mixing in the Standard Model

- ✓ Standard Model (SM) describes electromagnetic and weak interactions by a common gauge theory.
- ✓ The Lagrangian of the weak charged interactions of hadrons in the SM writes:

$$L_{\mathrm{int}\,J^cW} = \frac{g}{\sqrt{2}}\left(W_\alpha^{+} J^{C\alpha} + W_\alpha^{-} J^{C\alpha +}\right)$$

$J^{C\alpha} = \overline{U}_i V_{ik} \gamma^\alpha (1 \mp \gamma_5) D_k$ – hadronic charged current,

$g = e / \sin\theta_W$ – coupling constant,

θ_W – Weinberg angle,

$W_\alpha = W^{\pm}$ – charged boson fields

- ✓ The hadronic charged current links the vector of the *+2/3e* charged quarks *(u,c,t)* with the *-1/3e* charged quark vector *(d,s,b)* with the coupling constant V_{ik}.

Quark Mixing Matrix

- The weak eigenstates are not the flavour eigenstates of the strong interaction but their linear combination rotated by an angle θ. (First postulated by Cabibbo on the base of experimental data).
- For 2 quark doublets mixing is expressed via the real unitary matrix - rotation matrix on an angle θ_{Cab} (Cabibbo angle) in 2 dimensions.
- For 3 quark generations: the quark mixing is expressed via *3×3* unitary matrix *V* - Cabibbo-Kobayashi-Maskawa (CKM) matrix, which (by agreement) acts on the charge *–e/3* physical mass states and transforms them into new interaction eigenstates:

$$|q'_{Cab}\rangle = V|q\rangle = \begin{pmatrix} \cos\theta_{Cab} & \sin\theta_{Cab} \\ -\sin\theta_{Cab} & \cos\theta_{Cab} \end{pmatrix}$$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Experimental Values and Unitarity of the Quark Mixing Matrix

- In practice, the individual values of the entries of the mixing matrix can be determined on the base of the experimental data:

$$|V| = \begin{pmatrix} \text{[illegible]} & \text{[illegible]} & (\text{[illegible]} \pm \text{[illegible]}) \times \text{[illegible]} \\ \text{[illegible]} \pm \text{[illegible]} & \text{[illegible]} \pm \text{[illegible]} & (\text{[illegible]} \pm \text{[illegible]}) \times \text{[illegible]} \\ (\text{[illegible]} \pm \text{[illegible]}) \times \text{[illegible]} & (\text{[illegible]} \pm \text{[illegible]}) \times \text{[illegible]} & \text{[illegible]} \end{pmatrix}$$

- The unitarity of the CKM matrix – *unitary matrix* imposes

$$V_{us}^{\,\text{[illegible]}} + V_{ud}^{\,\text{[illegible]}} + \left|V_{ub}\right|^{\text{[illegible]}} = \text{[illegible]} \cong V_{us}^{\,\text{[illegible]}} + V_{ud}^{\,\text{[illegible]}} = \text{[illegible]} = \text{[illegible]}$$

- The unitarity check for the Cabibbo matrix:

$$V_{us}^{\,\text{[illegible]}} + V_{ud}^{\,\text{[illegible]}} = \text{[illegible]} = \text{[illegible]}$$

Wolfenstein Parameterisation of the Quark Mixing Matrix

- CKM matrix approximation in Wolfenstein parameters:

$$V=\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \cong \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3\left(1+\frac{\lambda^2}{2}\right)(\bar{\rho}-i\bar{\eta}) \\ -\lambda-iA^2\lambda^5\bar{\eta} & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\bar{\rho}-i\bar{\eta}) & -A\lambda^2-iA\lambda^4\bar{\eta} & 1 \end{pmatrix}$$

- Wolfenstein parameters: $\lambda, A, \bar{\rho}, \bar{\eta},$

CP violation:

$$\bar{\eta}\neq 0$$

$$\bar{\rho}+i\bar{\eta}=-\frac{V_{ub}^{*}V_{ud}}{V_{cb}^{*}V_{cd}}=\left|\frac{V_{ub}^{*}V_{ud}}{V_{cb}^{*}V_{cd}}\right|e^{i\gamma}$$

- Unitarity combined with experimental data yields values:

$$\lambda=0.2272\pm0.0010,\quad A=0.818^{+0.007}_{-0.017},$$

$$\bar{\rho}=0.221^{+0.064}_{-0.028},\quad \bar{\eta}=0.340^{+0.017}_{-0.045}$$

Standard Form of the CKM Matrix

- ✓ Parameterisation of the CKM - Physics does not depend on its choice, but we can choose most convenient one.
- ✓ Standard form of CKM parameterisation:

Simple form

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

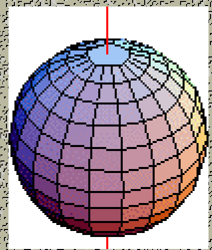
$$c_{ij} = \cos\theta_{ij}$$
$$s_{ij} = \sin\theta_{ij}$$
$$i,j = 1,2,3$$

- ✓ Three angles θ_{ij}; when $\theta_{23}=\theta_{13}=0$, the third $\theta_{12} = \theta_c$ - Cabibbo case.
- ✓ CKM matrix can be expressed as the product of 3 real rotation matrices R_{ij} and 3 diagonal matrices P1, P2, P3:

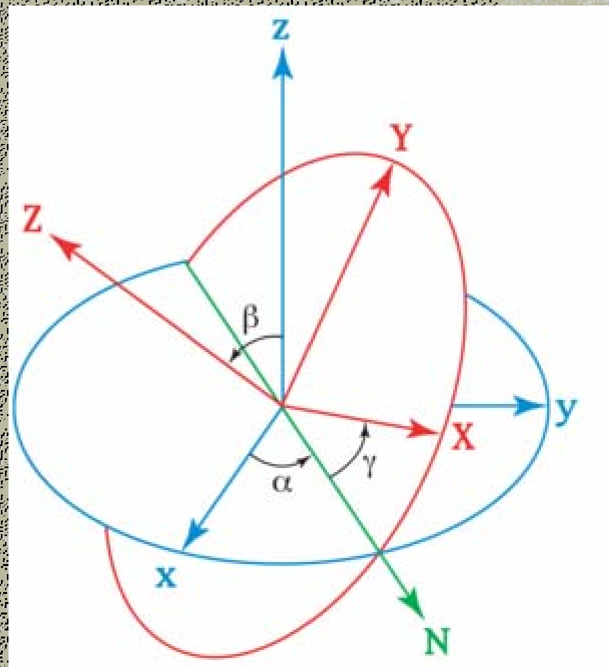
$$V = \mathbf{P}_2\mathbf{R}_{23}\mathbf{R}_{13}\mathbf{P}_1\mathbf{R}_{12}\mathbf{P}_3, \quad \mathbf{P}_1 = \begin{pmatrix} \square & \square & \square \\ \square & e^{-i\delta} & \square \\ \square & \square & \square \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} e^{-i\delta} & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}, \quad \mathbf{P}_3 = \begin{pmatrix} e^{i\delta} & \square & \square \\ \square & e^{i\delta} & \square \\ \square & \square & \square \end{pmatrix}.$$

Rotations in Euler Angles Form

- ✓ **The rotation M**, according to Euler's rotation theorem - **M=BCD**, where B,C and D are rotation matrices - is given by three rotation angles α, β, γ.



The example of the rotation in Euler Angles.
The xyz (fixed) system is shown in blue, the XYZ (rotated) system is shown in red. The line of nodes, labelled N, is shown in green



Standard Form of the CKM Matrix and the Pitch-Roll-Yaw Rotations

- ✓ Standard Form of the CKM matrix without the CP violation, accounted via complex phase δ, being real, represents the rotation **M=BCD** in the "*xyz*" – pitch-roll-yaw convention (common in aeronautics), where

roll: θ_{12} – rotation about the *x*-axis,
pitch: $-\theta_{13}$ rotation about the *y*-axis and
yaw: θ_{23} – rotation about the *z*-axis.

Y – to the right

X - forward

Z - down

Exponential Parameterisation of the CKM Matrix and Rotations

- ✓ Exponential form of mass mixing matrix: $\hat{V} = e^{\mathbf{\Lambda}}$,

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & \lambda & \alpha\lambda e^{i\delta} \\ -\lambda & 0 & -\beta\lambda \\ -\alpha\lambda e^{-i\delta} & \beta\lambda & 0 \end{pmatrix}$$

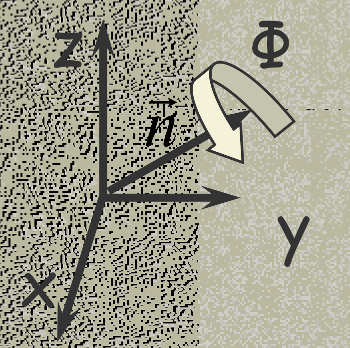
(δ accounts for the violation of CP, λ - for the quark mixing)

Classical rotation matrix M Single angle of rotation Φ and the direction unit vector

$$\mathbf{M}\left(\vec{\mathbf{n}}, \Phi\right) = e^{\Phi \mathbf{N}}$$

$$\hat{\mathbf{n}} = \left(n_x, n_y, n_z\right)$$

$$\mathbf{N} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$



- ✓ Mixing matrix in exponential parameterisation with conserved CP - $\delta=0$ - the angle-axis presentation of rotations in classical mechanics.

Standard form of CKM Matrix with δ=0 and Space Rotations in Angle-Axis

- ✓ Explicit expression for the rotation matrix in 3D space in the angle-axis form :

$$\mathbf{M}(\hat{\mathbf{n}},\Phi)=\begin{pmatrix} \cos\Phi+(1-\cos\Phi)n_x^2 & (1-\cos\Phi)n_x n_y-\sin\Phi n_z & (1-\cos\Phi)n_x n_z+\sin\Phi n_y \\ (1-\cos\Phi)n_y n_x+\sin\Phi n_z & \cos\Phi+(1-\cos\Phi)n_y^2 & (1-\cos\Phi)n_y n_z-\sin\Phi n_x \\ (1-\cos\Phi)n_z n_x-\sin\Phi n_y & (1-\cos\Phi)n_z n_y+\sin\Phi n_x & \cos\Phi+(1-\cos\Phi)n_z^2 \end{pmatrix}$$

- ✓ The rotation direction vector ***n*** and rotation angle **Φ** are related to the elements of the CKM matrix:

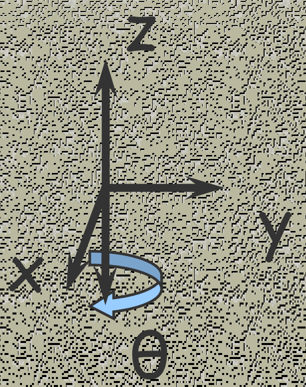
$$c_{ij}=\cos\theta_{ij} \qquad s_{ij}=\sin\theta_{ij}$$

$$n_x\sin\Phi=\left(-s_{23}\left(c_{12}+c_{13}\right)-s_{12}c_{23}s_{13}\right)/2 \qquad n_z\sin\Phi=\left(-s_{12}\left(c_{23}+c_{13}\right)-c_{12}s_{23}s_{13}\right)/2$$

$$n_y\sin\Phi=\left(s_{13}\left(1-c_{12}c_{23}\right)-s_{12}s_{23}\right)/2 \qquad \cos\Phi=\frac{1}{2}\left(c_{12}c_{13}+c_{12}c_{23}+c_{13}c_{23}-s_{12}s_{13}s_{23}-1\right)$$

Standard form of CKM Matrix with δ=0 and Space Rotations in Angle-Axis

- ✓ When two of the three angles of the CKM matrix turn zero, the last non vanishing angle becomes Φ.
- ✓ Cabibbo case takes place when the unit vector is turned against the *z*-axis:



$$\hat{\vec{n}}_{Cabibbo} = (0,0,-1)$$

$$V_{Cabibbo} = \exp\begin{pmatrix} \square & \lambda & \square \\ -\lambda & \square & \square \\ \square & \square & \square \end{pmatrix}$$

Exponential Mass Mixing Matrix with δ=0 and Space Rotations in Angle-Axis

- ✓ Comparison of the exponent of the rotation matrix $M(n,\Phi)$ in 3D with the exponential mass mixing matrix $V=e^{\Lambda}$, yields Φ and n

$$n_y = \alpha\lambda^2 \Big/ \sqrt{1+\beta^2\lambda^2+\alpha^2\lambda^4}$$

$$n_x = \beta\lambda \Big/ \sqrt{1+\beta^2\lambda^2+\alpha^2\lambda^4}$$

$$n_z = -1 \Big/ \sqrt{1+\beta^2\lambda^2+\alpha^2\lambda^4}$$

$$\Phi = \lambda\sqrt{1+\beta^2\lambda^2+\alpha^2\lambda^4}$$

- ✓ If we set the rotation angle in the axis-angle presentation $\Phi=0$, then we obtain the unit matrix I for the rotation $M(n,\Phi)$.

$$M(\vec{n},\Phi)\big|_{\Phi=0} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Quark Mixing with conserved CP and Rotations in Angle-Axis

- ✓ Thus we may imagine quark mixing when the CP is conserved as the rotation in the plane with the normal vector n, which determines the quark mixing by the single rotation angle, as the Cabibbo angle does for four quarks.

- ✓ It is enough to set this single fundamental angle zero in order to cancel the rotation from the geometric point of view and cancel the mixing between quarks from the physical point of view.

CP Violating Term and Rotations in Space

- ✓ The CP violation is accounted for via the complex phase in the entries (1,3) and (3,1) of the generating matrix *A*. Hence, the correspondent entries (1,3) and (3,1) of the rotation generator *M* should be complex:

$$\mathrm{V} = \exp\begin{pmatrix} \square & \lambda & \boxed{\alpha\lambda^{\square} e^{i\delta}} \\ -\lambda & \square & -\beta\lambda^{\square} \\ \boxed{-\alpha\lambda^{\square} e^{-i\delta}} & \beta\lambda^{\square} & \square \end{pmatrix} \Leftrightarrow M\left(\hat{\vec{n}}, \Phi\right) = \exp\left(\begin{bmatrix} 0 & -\Phi n_z & \boxed{\Phi n_y} \\ \Phi n_z & 0 & -\Phi n_x \\ \boxed{-\Phi n_y} & \Phi n_x & 0 \end{bmatrix}\right)$$

- ✓ Thus the "*y*" component of rotation direction vector ***n*** is complex:

$$\vec{\boldsymbol{n}} = \left(n_x, \mathrm{Re}\left[n_y\right], \mathrm{Im}\left[n_y\right], n_z\right)$$

Generation of New Parameterisations 1

- ✓ Let's decompose the argument of the mass mixing matrix $V=e^{A}$ in the sum of the Im[A] and Real[A] components: $A=A_1+A_2$

$$A_1=\tilde{A}_1=i\alpha\lambda^{3}\sin\delta\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$A_2=\tilde{A}_2=\begin{pmatrix}0&\lambda&\alpha\lambda^{3}\cos\delta\\-\lambda&0&-\beta\lambda^{2}\\-\alpha\lambda^{3}\cos\delta&\beta\lambda^{2}&0\end{pmatrix}$$

Make use of the identity from the matrix theory. The commutator is of the order of $O(\lambda^6)$, the last term $o(\lambda^6)$:

$$\hat{V}=e^{\hat{A}}=e^{\hat{A}_1+\hat{A}_2}=\exp\left(\frac{\hat{A}_1}{2}\right)\exp\left(\hat{A}_2\right)\exp\left(\frac{\hat{A}_1}{2}\right)+\frac{1}{24}\left[\hat{A}_1\left[\hat{A}_1\hat{A}_2\right]\right]+o\left(\lambda^{6}\right)$$

Generation of New Parameterisations 1

- ✓ Generate new exactly unitary parameterisation V with the help of above mentioned identity as follows:

$$\boxed{\hat{\mathbf{V}} = \mathbf{P}_{\mathbf{CP}} \cdot \mathbf{P} \cdot \mathbf{P}_{\mathbf{CP}},}$$

$$\mathbf{P}_{\mathbf{CP}} = \exp\left(\frac{\hat{A}}{\ }\right), \qquad \mathbf{P} = \exp\left(\hat{A}\right)$$

$$\mathbf{P} = V\big|_{\mathrm{Re}[A]} = \exp(\mathrm{Re}[A])$$

$$\mathbf{P}_{\mathbf{CP}} \Rightarrow \tilde{\mathbf{P}}_{\mathbf{CP}} = \begin{pmatrix} \cos\tilde{\Lambda} & & i\sin\tilde{\Lambda} \\ & & \\ i\sin\tilde{\Lambda} & & \cos\tilde{\Lambda} \end{pmatrix}$$

$$\tilde{\Lambda} = -\Lambda\sin\delta, \qquad \Lambda = \alpha\lambda,$$

Quark Mixing and Space Rotations

Generation of New Parameterisations 2

- The other decomposition of $A = A_1 + A_2$:

$$\mathbf{A}_2 = |\mathbf{A}| = \begin{pmatrix} 0 & \lambda & \alpha\lambda^3 \\ -\lambda & 0 & -\beta\lambda^2 \\ -\alpha\lambda^3 & \beta\lambda^2 & 0 \end{pmatrix}$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & \alpha\lambda^3\left(-1+e^{i\delta}\right) \\ 0 & 0 & 0 \\ \alpha\lambda^3\left(1-e^{-i\delta}\right) & 0 & 0 \end{pmatrix}$$

- Note: $e^{A_2} = P_{Rot}$ - rotation matrix in angle-axis presentation

Make use of the formula:

$$e^{\mathbf{A}_1+\mathbf{A}_2} \cong e^{\mathbf{A}_2} e^{\mathbf{A}_1}\left(1 - \frac{1}{2}\left[\mathbf{A}_2, \mathbf{A}_1\right]\right)$$

Generation of New Parameterisations 2

- ✓ Generate new parameterisation with the help of above mentioned identity with rotation matrix term:

$$\hat{\mathbf{V}} \cong \mathbf{P}_{\mathbf{Rot}}\mathbf{P}_{\mathbf{CP}}\left(\square - \frac{\square}{\square}\left[\mathbf{A}_2, \mathbf{A}_1\right]\right)$$

$$\mathbf{P}_{\mathbf{Rot}} = e^{\mathbf{A}\square} = \exp\begin{pmatrix} \square & \lambda & \alpha\lambda^{\square} \\ -\lambda & \square & -\beta\lambda^{\square} \\ -\alpha\lambda^{\square} & \beta\lambda^{\square} & \square \end{pmatrix}$$

$$\mathbf{P}_{\mathbf{CP}} = e^{\mathbf{A}\square} = \begin{pmatrix} \cos\square\Delta & \square & \kappa^{+}\sin\square\Delta \\ \square & \square & \square \\ \kappa^{-}\sin\square\Delta & \square & \cos\square\Delta \end{pmatrix}$$

$$\kappa^{\pm} = i e^{\pm i\frac{\delta}{\square}}$$

$$\Lambda = \alpha\lambda^{\square}$$

$$\Delta = \Lambda\sin\frac{\delta}{\square},$$

- ✓ The commutator is of the order $O(\lambda^4)$, then $\hat{\mathbf{V}}$ differs from $V=e^A$ in $o(\lambda^4)$.

Quark Mixing and Space Rotations

Comments on the Generation of New Parameterisations 2

- ✓ CP violating matrix reminds the Cabibbo mixing matrix for the 4 quarks, acting on the quarks *d* and *b* with the weights *κ* for the entries (1,3) and (3,1):

$$d' = d\cos\Delta + \kappa^{+} b \sin\Delta$$

$$b' = \kappa^{-} d \sin\Delta + b\cos\Delta$$

- ✓ Transformation P_{CP} preserves norm and orthogonality:

$$\langle d'|d'\rangle = \langle b'|b'\rangle = 1 \qquad \langle d'|b'\rangle = \langle b'|d'\rangle = 0$$

- ✓ Hermitian-conjugated and inverse with respect to P_{CP} exist:

$$\mathbf{P}_{\mathbf{CP}}^{-1}\cdot\mathbf{P}_{\mathbf{CP}} = \mathbf{P}_{\mathbf{CP}}^{+}\cdot\mathbf{P}_{\mathbf{CP}} = \mathbf{I}$$

Generation of New Parameterisations 3

- The generation of the new parametrisation with rotation matrix term with the help of formula from matrix theory:

$$\hat{\mathbf{V}} = e^{\hat{\mathbf{A}}} = \exp\left(\hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_2\right) = \exp\left(\frac{\hat{\mathbf{A}}_1}{\square}\right)\exp\left(\hat{\mathbf{A}}_2\right)\exp\left(\frac{\hat{\mathbf{A}}_1}{\square}\right) + \frac{\square}{\square}\left[\hat{\mathbf{A}}_1\left[\hat{\mathbf{A}}_1\hat{\mathbf{A}}_2\right]\right] + o\left(\lambda^{\square}\right)$$

$$\hat{\tilde{\mathbf{V}}} = \overline{\mathbf{P}}_{\mathbf{CP}} \cdot \mathbf{P} \cdot \overline{\mathbf{P}}_{\mathbf{CP}} \qquad \overline{\mathbf{P}}_{\mathbf{CP}} = e^{\frac{\Lambda}{\square}}, \ \mathbf{P} = \mathbf{P}_{\mathbf{Rot}} = e^{\Lambda\square}$$

$$\overline{\mathbf{P}}_{\mathbf{CP}} = \begin{pmatrix} \cos\Delta & \square & \kappa^{+}\sin\Delta \\ \square & \square & \square \\ \kappa^{-}\sin\Delta & \square & \cos\Delta \end{pmatrix}, \qquad \kappa^{\pm} = \pm i e^{\pm i\frac{\delta}{\square}}, \quad \Lambda = \alpha\lambda^{\square}, \quad \Delta = \Lambda\sin\frac{\delta}{\square}.$$

- P =P$_{rot}$ – rotational matrix, P$_{CP}$ – contains Imaginary part.

- $\tilde{\mathbf{V}}$ – new exactly unitary matrix, differs from V=e^{A} in $o(\lambda^9)$

Bessel function presentation of the New Parameterisations

- ✓ CP terms of the new parameterisations, expressed in Bessel Functions with distinguished CP phase δ:

$$\overline{\mathbf{P}}_{\mathbf{CP}} = \begin{pmatrix} \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\cos(\quad) & \kappa^{+} \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\sin(\quad) \\ \kappa^{-} \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\sin(\quad) & \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\cos(\quad) \end{pmatrix}$$

$$\mathbf{P}_{\mathbf{CP}} = \begin{pmatrix} \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\cos(\quad) & \kappa^{+} \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\sin(\quad) \\ \kappa^{-} \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\sin(\quad) & \sum\limits_{n=-\infty}^{\infty} J_n(\Lambda)\cos(\quad) \end{pmatrix}$$

Discussion and results

- ✓ Analogue between quark mixing in Standard Model with conserved CP and rotations in classical mechanics. Exponential parameterisation - rotation around a fixed axis in 3D space on angle Φ. When $\Phi = 0$, the mixing between quarks fades out since the mixing matrix becomes **I**.
- ✓ CP violation case is analogous to the rotation around axis with complex coordinate y.
- ✓ The exact treatment of problems with CP violation can be done on the base of theories with extended symmetries instead of O(3), for example SU(3) with Gell-Mann matrices, but the analysis become cumbersome!
- ✓ Generated new unitary parameterisations with CP violating part, distinguished in coefficients of expansion in Bessels.
- ✓ These new parameterisations include $O(\delta^9)$ corrections and abundantly satisfy experimental data confidence.