Semiclassical solyton type solutions of the nonlocal Gross-Pitaevsky equation

V. V. Belov, E. I. Smirnova
Department of applied mathematics,
Moscow State Institute of Electronics and Mathematics

[1] Belov V. V., Shapovalov A. V., Trifonov A. Yu. Teor. Mat.Phys
Semiclassical trajectory-coherent approximation
for the Hartree equation, 2002

[2] Belov V. V., Smirnova E. I. Localized asymptotic solutions
of an equation of an equation of a self-consistent
field. Mat. Zam., 80, 2, 2006

[3] Belov V. V., Kondrateva M. F., Smirnova E. I. Semiclassical soliton
type solutions of the Hartree equations., (to be published in DAN)

Nonstationary multidimensional Gross-Pitaevsky equation

$$-ih\frac{\partial}{\partial t}\Psi + \hat{H}_\varkappa \Psi = 0, \quad \|\Psi\|_{\mathbb{L}_2(\mathbb{R}^n_x)} = 1, \tag{1}$$

$$\hat{H}_\varkappa = \hat{H}_\varkappa[\Psi] = \hat{H}_0 + \varkappa \int_{\mathbb{R}^n} |\Psi(y,t)|^2 V(x-y)\, dy,$$

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + U(x),\ \hat{p} = -ih\nabla. \tag{2}$$

$\varkappa = \mathrm{Const}$, h-small parametr, $h \to 0$.
$U(x)$ is the smooth potential of the external field,
$V(x-y)$ is the smooth translate-invariant potential of the self-action.

Statment of problem: semiclassicaly soliton-type solutions , $h \to 0$

$$\begin{cases} -ih\frac{\partial}{\partial t}\Psi + \hat{H}_\varkappa[\Psi]\Psi = O(h^{3/2}), \quad (O(h^{3/2}) \text{ in the norm of } \mathbb{L}_2(\mathbb{R}^n_x)) \\ \Psi|_{t=0} = \Psi_0(x,h) = \exp[\frac{i}{h}((p_0, x-x_0) + (x-x_0, B_0(x-x_0)))] \end{cases} \tag{3}$$

$B_0 = B_0^t$, $\mathrm{Im} B_0 > 0$

In the linear case $(\varkappa = 0)$ in (2),$\hat{H}_\varkappa = \hat{H}_0 = \frac{\hat{p}^2}{2} + U(x)$ asymptotic solutions $(h \to 0)$ have the folowing form:

$$\Psi(x,t,h) = N\exp[\frac{i}{h}(S_{\text{cl}}(t) + (p_{\text{cl}}(t), x - x_{\text{cl}}(t))]\cdot$$
$$\cdot\exp[\frac{i}{h}(x - x_{\text{cl}}(t), BC^{-1}(t)(x - x_{\text{cl}}(t)))](\text{detC(t)})^{-1/2}. \quad (1)$$

Where $S_{\text{cl}}(t)$ is the classical action

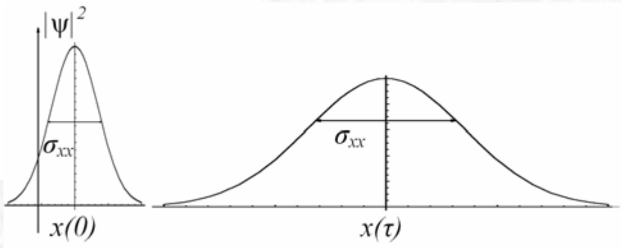
$$S_{\text{cl}} = \int_0^t \left(\frac{\dot{x}_{\text{cl}}^2(\tau)}{2} - U(x_{\text{cl}}(\tau))\right) d\tau, \quad (2)$$

system in variations

$$\begin{pmatrix}\dot{B}\\ \dot{C}\end{pmatrix} = \begin{pmatrix}0 & -U''_{xx}(x_{\text{cl}}(t))\\ \mathbb{E}_n & 0\end{pmatrix}\begin{pmatrix}B\\ C\end{pmatrix},$$
$$B(0) = B_0, C(0) = \mathbb{E}_n = ((\delta_{ij}))_{n\times n}. \quad (3)$$

Classical system $\begin{cases}\dot{p}_{\text{cl}} = \frac{\partial U(x_{\text{cl}})}{\partial x_{\text{cl}}}\\ \dot{x}_{\text{cl}} = p_{\text{cl}}\\ p_{\text{cl}}|_{t=0} = p_0,\ x_{\text{cl}}|_{t=0} = x_0\end{cases}$ - apriory Hamiltonian system

$$\sigma_{xx}(t) = \langle\Psi, (x - x_{cl})^2\Psi\rangle_{\mathbb{L}_2} \to \infty,\ t \to \infty$$



The first main questions arises: What is classical dynamical system corresponding to nonlinear $\hat{H}_\varkappa$ in the limit $h \to 0$?

Classical trajectory of quantum particle in the state $\Psi(x,t,h) : ih\Psi_t = \hat{H}_\varkappa \Psi$

$Z_\Psi(t,h) = (P(t,h), X(t,h) \in \mathbb{R}^{2n}, P(t,h) = \langle \Psi, -ih\nabla\Psi\rangle_{\mathbb{L}_2}, X(t,h) = \langle \Psi, x\Psi\rangle_{\mathbb{L}_2})$.

$P(t,h), X(t,h)$ are smooth functions, $h \to 0$.

Statment 1. Classical system corresponding to the nonlinear operator $\hat{H}_\varkappa$(Hamilton -Erenfest system):

$$\begin{cases} \dot{p} = -\nabla_x U(x) - \varkappa\nabla_\tau V(0) - \\ \qquad -\nabla_x[\frac{1}{2}\mathrm{Sp}(\mathrm{U}''_{\mathrm{xx}}(\mathrm{x}) + 2\varkappa \mathrm{V}''_{\mathrm{xx}}(\mathrm{x}-\mathrm{y}))\sigma_{\mathrm{xx}})]|_{\mathrm{y}=\mathrm{x}} \\ \dot{x} = p \\ \dot{\Delta} = JM_\varkappa(x)\Delta - \Delta M_\varkappa(x)J, \quad (p,x) \in \mathbb{R}^{2n}. \end{cases} \qquad (1)$$

$$\Delta = \begin{pmatrix} \sigma_{pp} & \sigma_{px} \\ \sigma_{xp} & \sigma_{xx} \end{pmatrix} \text{ – are of order } O(h), \sigma^t_{xp} = \sigma_{px}, \ \sigma^t_{xx} = \sigma_{xx}, \ \sigma^t_{pp} = \sigma_{pp},$$

$$M_\varkappa(x) = \begin{pmatrix} \mathbb{E}_n & 0 \\ 0 & U''_{xx}(x) + \varkappa V''_{\tau\tau}(0) \end{pmatrix}, \ U''_{xx}(x) = \left(\frac{\partial^2 U(x)}{\partial x_i \partial x_j}\right)_{n\times n},$$

Initial conditions

$$\begin{cases} p|_{t=0} = \langle \Psi_0, \hat{p}\Psi_0\rangle = p_0, \ x|_{t=0} = \langle \Psi_0, x\Psi_0\rangle = x_0 \\ (\sigma_{pp}|_{t=0})_{km} = \langle \Psi_0, \hat{\Delta}p_k\hat{\Delta}p_m\Psi_0\rangle = \frac{h}{4}(B_0B_0^+ + B_0^*B_0^t) \\ (\sigma_{xx}|_{t=0})_{km} = \langle \Psi_0, \hat{\Delta}x_k\hat{\Delta}x_m\Psi_0\rangle = \frac{h}{4} \\ (\sigma_{px}|_{t=0})_{km} = \frac{1}{2}\langle \Psi_0, (\hat{\Delta}p_k\hat{\Delta}x_m + \hat{\Delta}x_m\hat{\Delta}p_k)\Psi_0\rangle = \frac{h}{4}(B_0 + B_0^t), \ k,m = \overline{1,n} \end{cases}$$

Hamiltonian system in variations with self- action:

$$\begin{pmatrix} \dot{B} \\ \dot{C} \end{pmatrix} = JM_{\varkappa}(X_0(t)) \begin{pmatrix} B \\ C \end{pmatrix}, \quad B|_{t=0} = B_0,\ C|_{t=0} = \mathbb{E}_n \qquad \Delta(t) = A(t)\Delta|_{t=0}A^{+}(t) \tag{1}$$

$A(t)$ -the Caushy matrix

where X_0- is the main part of the $X(t,h) = X_0(t) + hX_1(t) + O(h^2)$

Remark: $A(t) = \begin{pmatrix} B(t) & B^*(t) \\ C(t) & C^*(t) \end{pmatrix} \begin{pmatrix} B(0) & B^*(0) \\ C(0) & C^*(0) \end{pmatrix}^{-1}$

Statement

Let the folowing conditions to be satisfy: $U(x), V(\tau) \in C^3(\mathbb{R}^n)$. Then localized asymptotic solution $\pmod{h^{3/2}}$, $h \to 0$, $t \in [0,T]$ has the form

$$\Psi_\nu(x,t,h) = N_\nu \exp[\frac{i}{h}(S_{\varkappa}(t,h) + (P(t,h), x - X(t,h)))]$$
$$\exp[\frac{i}{2h}(x - X(t,h), BC^{-1}(t)(x - X(t,h)))]\frac{1}{\sqrt{\mathrm{detC(t)}}}.$$

where $S_{\varkappa}(t,h)$ is the classical phase

$$S_{\varkappa}(t,h) = \int_0^t \left(\frac{\dot{X}^2}{2}(\tau,h) - U(X(\tau,h))\right) d\tau - \varkappa V(0)t - \frac{\varkappa}{2}\int_0^t \mathrm{Sp}(\mathrm{V}''_{\tau\tau}(0)\sigma_{\mathrm{xx}}(\tau,\mathrm{h}))\mathrm{d}\tau.$$

Semiclassical solyton type solutions $(U(x)=0)$

Let's determine ω_j^2, $(j=1,...,n)$-eigen values of matrix $\varkappa V''_{\tau\tau}(0)$

Theorem:Let the folowing conditions to be satisfied: $\varkappa V''_{\tau\tau}(0)>0$, $V(\tau)\in C^3(\mathbb{R}^n)$ and $\omega_j^2\neq\omega_k^2$, $j\neq k$, then semiclassical solyton type solutions has the form:

$$\Psi\ (x,t,h)=N_\nu\exp[\frac{i}{h}(S_\varkappa(t,h)+(P(t,h),x-X(t,h)))]$$
$$\exp[\frac{i}{2h}(x-X(t,h),\dot{C}C^{-1}(t)(x-X(t,h)))]\frac{1}{\sqrt{C(t)}}, \qquad (1)$$

$$S_\varkappa(t,h)=\int_0^t\frac{\dot{X}^2}{2}(\tau,h)\,d\tau-\varkappa V(0)t-\frac{\varkappa}{2}\int_0^t\mathrm{Sp}(V''_{\tau\tau}(0)\sigma_{xx}(\tau,h))\,d\tau.$$

$$X(t,h) = x_0 + p_0 t + \frac{(a_0 + h a_1)t^2}{2} +$$
$$+ h \sum_{\pm} \sum_{l,m=1, l\neq m}^{n} (d^{\pm}_{l,m} \cos((\omega_l \pm \omega_m)t) + k^{\pm}_{l,m} \sin((\omega_l \pm \omega_m)t)) +$$
$$+ h \sum_{l=1}^{n} (e_l \cos((2\omega_l)t) + s_l \sin(2(\omega_l)t)),$$
$$P(t,h) = \dot{X}(t,h), \quad a_0 = -\varkappa \nabla_\tau V(0),$$
$$\sigma_{xx}(t,h) = h\Gamma_0 + \sum_{\pm} \sum_{l,m=1, l\neq m}^{n} (\cos((\omega_l \pm \omega_m)t) A^{\pm}_{l,m} -$$
$$- \sin((\omega_l \pm \omega_m)t) B^{\pm}_{l,m}) + \sum_{l=1}^{n} A^{+}_{l,l} \cos(2\omega_l t) - B^{+}_{l,l} \sin(2\omega_l t),$$
$$||\sigma_{xx}(t)|| < \mathrm{Const}, \ \forall \mathrm{t} \in \mathbb{R}$$

where Γ_0, $A^{\pm}_{l,m}$, $B^{\pm}_{l,m}$ $-n \times n$ is the constant real matrix, a_1, $d^{\pm}_{l,m}$, $k_{,l,m}$, s_l, e_l- real vectors from $\mathbb{R}^n$, $l, m = 1, \ldots, n$ explicit form is easily calculated with formula $\Delta(t,h) = A(t)\Delta|_{t=0} A^{+}(t)$ whith the Caushi matrix $A =$

$$\begin{pmatrix} \cos[t\sqrt{R}] & R^{-1/2} \sin[t\sqrt{R}] \\ -\sqrt{R} \sin[t\sqrt{R}] & \cos[t\sqrt{R}] \end{pmatrix}, \ R = \varkappa V''_{\tau\tau}(0).$$

$(U(x) = < E(t), x >)$:

Let be $V(\tau)$ –smooth function, $\tau \in R^3$, grow with all it's derivates at $|\tau| \to \infty$ not faster, than $Ce^{|\tau|}, C = Const > 0$, and $\nabla_\tau V(0) = 0$, $\kappa V''_{\tau\tau}(0) > 0$. Then semiclassical solyton type solutions $\mod h^{3/2}$ of the equation (1) in non-stationary homogeneous electric field $(U(x) = < E(t), x >, E(t) \in R^3)$ have the form

$$\Psi(x,t,h) = \exp[-\frac{i}{h}\lambda_\nu t]\exp[\frac{i}{h}\int_0^t L(X(\tau), \dot{X}(\tau), \tau)d\tau]\exp[\frac{i}{h} < \dot{X}(t)m, x - X(t) >]\Psi_{\lambda_\nu}(x - X(t))$$

where $L(X(t), \dot{X}(t), t) = \frac{< m\dot{X}(t), \dot{X}(t) >}{2} - < E(t), X(t) >$,

$X(t)$-centre of gravity of the semiclassical solyton (-the any solution of the equation of Newton)– $m\ddot{X}(t) = -E(t)$,

$(\lambda_\nu, \Psi_{\lambda_\nu}(x)), x \in R^3$- E. values and E.- function operator $\frac{(\hat{p})^2}{2m} + \kappa \int_{R^3} V(x-y)\,|\Psi|^2 dy$ ·

$\lambda_\nu, \nu = (\nu_1, \nu_2, \nu_3), \nu_j \in Z$:

$$\lambda_\nu = \kappa V(0) + h\sum_{k=1}^{3} \Omega_k (\nu_k + \frac{1}{2}) + \frac{1}{2} Sp[V_{\tau\tau}(0)\sigma_{xx}],$$

$$\sigma_{xx} = \frac{h}{4}(CD_\nu C^+ + C^* D_\nu C^T)$$

where 3×3 matrix C consist of the E. vectors $f_j, < f_j, f_{j'} >= \delta_{j'j}, j, j' = 1,2,3$, which associate E. values $\Omega_j^2, j = 1,2,3$

of the matrix $\kappa V''_{\tau\tau}(0)$. $D_\nu = diag(2\nu_k + 1), k = 1,2,3$.

$$\Psi_{\lambda_\nu}(x,h) = (\pi h)^{-3/4} (\det C)^{-1/2} \exp[\frac{i}{h}\frac{1}{2} < \Delta x, BC^{-1}\Delta x >] He_\nu\, (-i(\tilde{C}^*)^{-1} \frac{\Delta x\sqrt{2}}{\sqrt{h}})$$

$He_\nu(\xi), \xi \in R^3$ - multidimentional Hermit polynomial, which associate matrix $W = -\tilde{C}^+(\tilde{C}^{-1})^T$,

$(He_\nu(\xi) = \frac{(-1)^{|n|}}{\sqrt{n!}}(\frac{\partial}{\partial \xi} - W\xi)^n \cdot 1)$, $\Delta x = x - x_0, x_0 \in R^3$

Definition. **Class of trajectory-concentrated functions**

$Z(t,h) = (P(t,h), X(t,h)) \in \mathbb{R}^{2n}$, $S(t,h) \in \mathbb{R}, S(0,h) = 0$

$$\forall t \, K = \{\Phi, \Phi = \phi(\frac{x - X(t,\hbar)}{\sqrt{\hbar}}, t)) \exp[\frac{i}{\hbar}(S(t,\hbar) + (P(t,\hbar), x - X(t,\hbar)))]\},$$

where $\phi(\xi, t) \in S(\mathbb{R}^n)$

Assuption: Let $\Psi(x,t,h)$ exist and satisfy eq. Hartre $\pmod{h^{3/2}}$, $\Psi \in K)$

Example

$V(\tau) = \exp[-\sum_{i=0}^{3}\gamma_i(\tau_i,\tau_i)]$, $\tau_i \in \mathbb{R}^1$, $\gamma_i < 0$.

Solinon type solution has the form

$$\Psi_\nu(x,t,h) = N_\nu \exp[\frac{i}{h}(S_\varkappa(t,h) + (P(t,h), x - X(t,h)))]$$

$$\exp[\frac{i}{2h}(x - X(t,h), \dot{C}C^{-1}(t)(x - X(t,h)))]\frac{1}{\sqrt{C(t)}}\hat{\Lambda}^{+\nu}(t)1,$$

$$((C(t)))_{ij} = ((\frac{b_{ij}}{\Omega_i}\sin(t\Omega_i) + \delta_{ij}\cos(t\Omega_i)))_{ij}$$

$$S_\varkappa(t,h) = \int_0^t \frac{\dot{X}^2}{2}(\tau,h)\,d\tau - \varkappa\int_0^t \sum_{i=0}^{3}\gamma_i\sigma_{x_ix_i}(\tau,h))\,d\tau,$$

$$\sigma_{x_ix_i} = \frac{h(1+2\mu_i)}{8\mathrm{ImB}\Omega_{\mathrm{i}}^2}(|b|^2 + \Omega_i^2 + \cos(2t\Omega_i)(-|b|^2 + \Omega_i^2) + \sin(2t\Omega_i)2(\mathrm{Reb})^2\Omega_{\mathrm{i}})$$

$$\hat{\Lambda}^{+\nu} = [\hat{\Lambda}_1^+]^{\nu_1}\dots[\hat{\Lambda}_n^+]^{\nu_n},$$

$$\hat{\Lambda}_j^+(t) = \frac{1}{\sqrt{h}}((z_j^*(t),\hat{p}) - (\dot{z}_j^*(t) - \dot{C}C^{-1}(t)z_j^*(t),(x - X(t,h)))),$$

$z_j(t) -$ columns of matrix$C(t)$